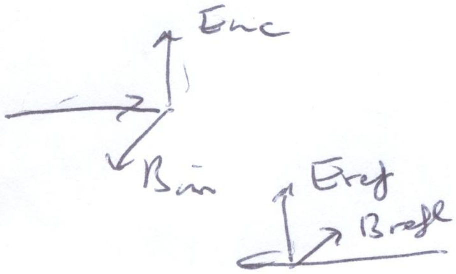
Lecture #20

P2a

ε 2.



$$\omega^2 = \frac{c^2 k^2}{c n}$$

$$E_{in} = \hat{y} \int_{-\infty}^{\infty} d\omega\, f(\omega)\, e^{i\frac{\omega}{c}(x - ct)}$$

$x = 0 \quad t < 0$ no wave

$\Rightarrow f(\omega)$ analytic in upper half pl.

$$E_{tr} = \hat{y} \int_{-\infty}^{\infty} d\omega\, A(\omega)\, e^{i\frac{\omega}{c}(x - ct)}$$

not wave for $x > 0$

$t < 0$

$\Rightarrow A'(\omega)$ is analytic in upper half pl.

$$E_r = \hat{y} \int_{-\infty}^{\infty} d\omega\, B(\omega)\, e^{i\frac{\omega}{c}(-x - ct)}$$

k vector is opposite

$x < 0 \quad t < 0$

no reflected wave when

$B(\omega)$ analytic in upper half plane

$$B(\omega) = -f(\omega) R(\omega)$$

$$A'(\omega) = T(\omega) f(\omega)$$

$$R(\omega) + T(\omega) = 1$$

$$B_{in} = \frac{c}{\omega} \vec{k} \times \hat{y} \int d\omega\, f(\omega)\, e^{i\omega(x - ct)}$$

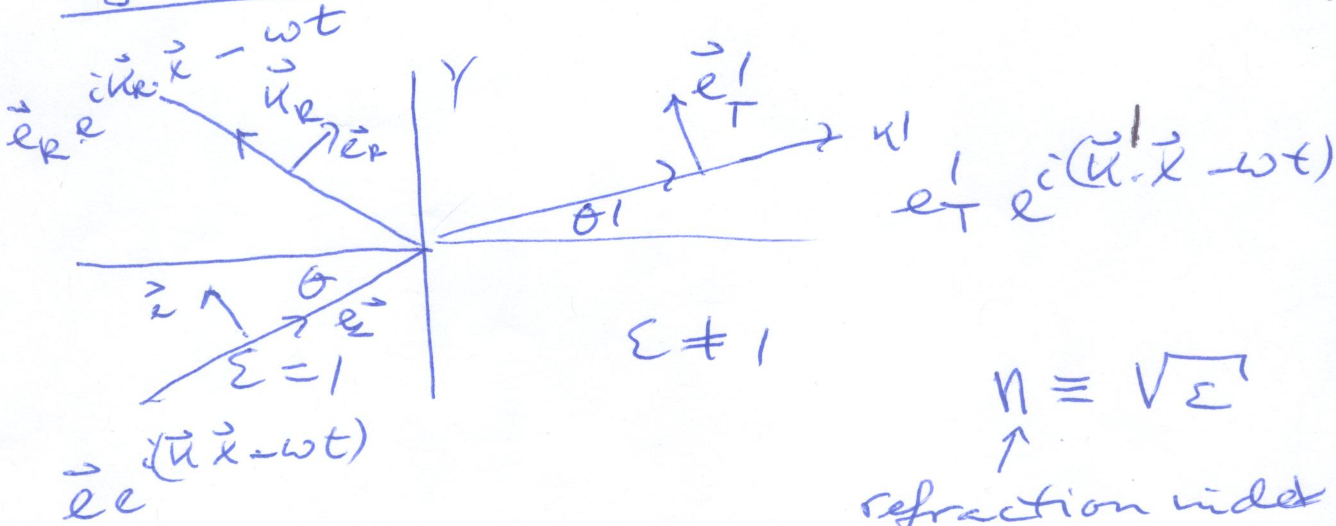
$$B_{ref} = -\frac{c}{\omega} \vec{k}' \times \hat{y} \int d\omega\, B(\omega)\, e^{i\omega(-x - ct)}$$

$$B_{tr} = \frac{c}{\omega} \vec{k}' \times y \int d\omega\, A'(\omega)\, e^{ik'x - i\omega t}$$

$$T(\omega) = \frac{2}{1 + \sqrt{\varepsilon}}$$

$$R(\omega) = \frac{\sqrt{\varepsilon} - 1}{\sqrt{\varepsilon} + 1}$$

Reflection and Refraction



$n \equiv \sqrt{\varepsilon}$

refraction index

The frequencies of both sides are the same

$$\vec{k}^2 = \vec{k}_R^2 = \frac{\omega^2}{c^2} = \frac{\vec{k}'^2}{\varepsilon}$$

$\vec{E}$ has only component in the xy plane

$$\Rightarrow k_x^2 + k_y^2 = \frac{(k_x'^2 + k_y'^2)}{\varepsilon}$$

boundary conditions – E_{tan} continuous
– B_{tan} continuous

They must hold at $x = 0$ in the full ~~y~~ z plane $\Rightarrow$ k_y and k_z must be continuous. There is no k_z by assumption

So we get $\frac{\sin\theta'}{\sin\theta} = \frac{\cancel{k_y}}{k'}\frac{k}{\cancel{k_y}} = \frac{k}{k'} = \frac{1}{\sqrt{\varepsilon}}$

$\Rightarrow \quad n \sin\theta' = \sin\theta$

This is <u>Snel's law</u>

Next we determine the full solution.

$x > 0 \qquad \vec{E} = \vec{e}_T \, e^{i(k_x' x + k_y y - \omega t)}$

$B = \frac{c\vec{k}'}{\omega} \times \vec{E}$

$x < 0 \qquad \vec{E} = \vec{e} e^{ik_x x + k_y y - \omega t}$

$- \vec{e}_R \, e^{i(-k_x x + k_y y - \omega t)}$

$\uparrow$ left moving wave

$B = \frac{c}{\omega} \vec{k} \times \vec{E}$

note that everywhere we have the same y-dependence.

$\vec{e} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \qquad \vec{e}_R = R \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}$

$\vec{e}_T = T \begin{pmatrix} -\sin\theta' \\ \cos\theta' \end{pmatrix}$

continuity of the tangential (y) component gives

$\cos\theta (1 - R) = T \cos\theta'$

Lecture # 27

(845)

Today

Helicity

Vector and scalar potential

Coulomb gauge

Lorentz gauge

Green's function

For the magnetic field which has only a $\hat{z}$ component we obtain

$$\vec{B}_z = \frac{c}{\omega}(\vec{k} \times \vec{E})_z$$

$$\vec{k} \perp \vec{E} \quad = \frac{c}{\omega}\hat{z}\,|k|\,(1+R) \quad \frac{c}{\omega}(k_x\cos\theta - k_y(-\sin\theta))$$

$$\vec{k}_R = (-k_x, k_y)$$

$$\vec{e}_R = R(\sin\theta, \cos\theta)$$

$$\vec{B}_z^R = \frac{c}{\omega}(k_{Rx}e_{Ry} - k_{Ry}e_{Rx})$$

$$= \frac{c}{\omega}R(-k_x\cos\theta - k_y\sin\theta)$$

$$= -\frac{cR}{\omega}(\vec{k} \times \vec{e})$$

($\vec{e}_R$ was defined with a minus sign)

for B_z of transmitted wave

$$B_z^t = \hat{z}\,\frac{c k'}{\omega}\,T$$

$$\Rightarrow \frac{ck}{\omega}(1+R) = \frac{ck'}{\omega}T$$

$$\Rightarrow k(1+R) = k'T$$

$$\cos\theta\,(1-R) = T\cos\theta'$$

$$\Rightarrow 2 = T\left(\frac{k'}{k} + \frac{\cos\theta'}{\cos\theta}\right)$$

$$\Rightarrow T = \frac{2k\cos\theta}{k'\cos\theta + k\cos\theta'}$$

$$R = \frac{k'}{k} T - 1$$

$$= \frac{k' 2 \cos\theta}{k' \cos\theta + k \cos\theta'} - 1$$

$$= \frac{k' \cos\theta - k \cos\theta'}{k' \cos\theta + k \cos\theta'}$$

$R + T = 1$ for $\theta = \theta' = 0$

Special cases

very good conductor $\quad \varepsilon(\omega) = \varepsilon + \frac{4\pi i \sigma}{\omega}$

$\sigma \to \infty \Rightarrow |\varepsilon| \gg 1$

$\Rightarrow k' \gg k$

$\Rightarrow R \to 1$

So we see why metals are shiny.

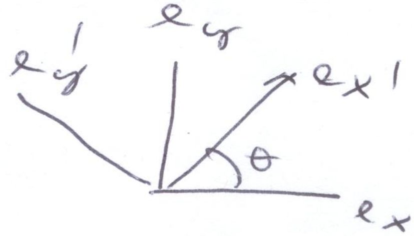
Brewster angle $\quad R = 0$ if $k' \cos\theta = k \cos\theta'$

$\Rightarrow \cos\theta = \frac{\sin\theta'}{\sin\theta} \cos\theta'$

$\Rightarrow \sin 2\theta = \sin 2\theta' \Rightarrow \theta' = \theta + \frac{\pi}{2}$

$\frac{1}{\sqrt{\varepsilon}} = \frac{\sin\theta'}{\sin\theta} = \frac{\sin(\theta + \frac{\pi}{2})}{\sin\theta} = \frac{\cos\theta}{\sin\theta} \Rightarrow \tan\theta = \sqrt{\varepsilon}$

Helicity



$$e_\pm = \frac{\hat{e}_x \pm i\hat{e}_y}{\sqrt{2}}$$

$$e_\pm' = \frac{\hat{e}_x' \pm i\hat{e}_y'}{\sqrt{2}}$$

$$\begin{pmatrix} e_x' \\ e_y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} e_x \\ e_y \end{pmatrix}$$

$$\Rightarrow e_\pm' = \frac{(\cos\theta\, \hat{e}_x + \sin\theta\, \hat{e}_y)}{\sqrt{2}} \pm i \frac{(-\sin\theta\, \hat{e}_x + \cos\theta\, \hat{e}_y)}{\sqrt{2}}$$

$$= (e^{\mp i\theta}\, \hat{e}_x \pm i\, \hat{e}_y\, e^{\mp i\theta}) \frac{1}{\sqrt{2}}$$

$$= e^{\mp i\theta}\, \vec{e}_\pm$$

So $\vec{e}_\pm$ have helicity $\pm$ by definition of helicity.

Imaginary ε

Good conductor $\quad \varepsilon = \frac{4\pi i \sigma}{\omega}$

$$K_y' = K_y$$

Frequency is continuous

$$k_x'^2 + k_y'^2 = (k_x^2 + k_y^2)\,\varepsilon = \frac{\omega^2 \varepsilon}{c^2}$$

$$\varepsilon \gg 1$$
$$k_y = k_y' \quad \Rightarrow \quad k_x' \gg k_x$$

$$\Rightarrow k_x' \approx \frac{\omega}{c}\sqrt{\varepsilon}$$
$$= \frac{\omega}{c}\sqrt{\frac{4\pi\sigma}{\omega}}\,\sqrt{i}$$

$$\Rightarrow E_T = \vec{e}_T \; e^{i K_y y + i x \frac{\omega}{c}\sqrt{\frac{2\pi\sigma}{\omega}}(1+i)}$$

$$\sqrt{i} = \frac{1+i}{\sqrt{2}}$$

damping factor $\quad e^{-xK\sqrt{\frac{2\pi\sigma}{\omega}}}$

penetration depth $\quad \frac{1}{K\sqrt{\frac{2\pi\sigma}{\omega}}}$

Continuity of the normal component of $\vec{D}$

$\vec{D}_n$ continuous

R has − sign

$x<0 \qquad D_n = -\sin\theta \mp R\sin\theta$

$x>0 \qquad D_n = -\varepsilon T\sin\theta'$

$\varepsilon = \dfrac{k'^2}{k^2} \Rightarrow (1+R)\sin\theta = \dfrac{k'^2}{k^2} T\sin\theta'$

Snel $\quad \dfrac{k'}{k}\sin\theta' = \sin\theta$

$\Rightarrow$

$(1+R)\sin\theta = \dfrac{k'}{k} T\sin\theta$

This is what we got from the continuity of B_z

Continuity of the normal component of $\vec{D}$

$\vec{D}_n$ continuous

$x<0$ $\quad D_n = -\sin\theta + R\sin\theta$

$x>0$ $\quad D_n = -\varepsilon T \sin\theta'$

$\varepsilon = \dfrac{k'^2}{k^2} \Rightarrow (1+R)\sin\theta = \dfrac{k'^2}{k^2} T \sin\theta'$

<u>Snel</u> $\quad \dfrac{k'}{k} \sin\theta' = \sin\theta$

$\Rightarrow$

$(1+R)\sin\theta = \dfrac{k'}{k} T \sin\theta$

This is what we got from the continuity of B_z

Vector and scalar potential

$\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow \exists \vec{A}$ such that $\vec{B} = \vec{\nabla}\times\vec{A}$

$$\Rightarrow \vec{\nabla}\times\vec{E} + \frac{1}{c}\partial_t \vec{\nabla}\times\vec{A} = 0$$

$$\Rightarrow \vec{\nabla}\times\left(\vec{E} + \frac{1}{c}\partial_t \vec{A}\right) = 0$$

$\Rightarrow \exists$ scalar potential ϕ

$$\vec{E} + \frac{1}{c}\partial_t \vec{A} = -\vec{\nabla}\phi$$

$$\Rightarrow \vec{B} = \vec{\nabla}\times\vec{A}$$
$$\vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\partial_t \vec{A}$$

gauge invariant $\quad \vec{A} \to \vec{A} + \vec{\nabla}\chi$

$$\phi \to \phi - \frac{1}{c}\partial_t \chi$$

Inhomogeneous Maxwell equations in terms of gauge potential

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho \Rightarrow -\vec{\nabla}^2\phi - \frac{1}{c}\vec{\nabla}\partial_t\vec{A} = 4\pi\rho$$

$$\vec{\nabla}\times\vec{B} - \frac{1}{c}\partial_t\vec{E} = \frac{4\pi}{c}\vec{J} \Rightarrow$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) + \frac{1}{c}\partial_t\vec{\nabla}\phi + \frac{1}{c^2}\partial_t^2\vec{A} = \frac{4\pi}{c}\vec{J}$$

We can simplify these equations by a suitable gauge choice

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \vec{\nabla}^2 A$$

Helicity

Lecture #20

$$-\nabla^2\phi - \frac{1}{c}\partial_t \vec{\nabla}\vec{A} = 4\pi\rho$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) + \frac{1}{c}\partial_t\vec{\nabla}\phi + \frac{1}{c^2}\partial_t^2\vec{A} = \frac{4\pi}{c}\vec{j}$$

Coulomb gauge $\vec{\nabla}A = 0$

$$\Rightarrow \vec{\nabla}^2\phi = -4\pi\rho$$

$$-\vec{\nabla}^2 A + \frac{1}{c^2}\partial_t^2\vec{A} = \frac{4\pi}{c}j - \frac{1}{c}\partial_t\vec{\nabla}\phi$$

$$\nabla(\text{r.hs}) = \frac{4\pi}{c}\vec{\nabla}j - \frac{1}{c}\partial_t\vec{\nabla}^2\phi$$

$$= -\frac{4\pi}{c}\partial_t\rho + \frac{4\pi}{c}\partial_t\rho = 0$$

gauge invariance $\vec{A} \to \vec{A} + \vec{\nabla}\chi$

$$\phi \to \phi - \frac{1}{c}\partial_t\chi$$

Today Lorentz gauge

Green's function

Coulomb gauge $\qquad \vec{\nabla}\cdot\vec{A}=0$

$$\Rightarrow \quad -\vec{\nabla}^2\vec{A}+\frac{1}{c^2}\partial_t^2\vec{A}=\frac{4\pi}{c}\vec{j}-\frac{1}{c}\partial_t\vec{\nabla}\phi$$

$$\vec{\nabla}^2\phi=-4\pi\rho$$

instantaneous Coulomb potential

The rhs is transverse:

$$\vec{\nabla}\cdot\left(\frac{4\pi}{c}\vec{j}-\frac{1}{c}\partial_t\vec{\nabla}\phi\right)$$

$$=\frac{4\pi}{c}\vec{\nabla}\cdot\vec{j}-\frac{1}{c}\partial_t\vec{\nabla}^2\phi=\frac{4\pi}{c}\underbrace{(\vec{\nabla}\cdot\vec{j}+\partial_t\rho)}_{=0}$$

by continuity eq.

longitudinal component of $\vec{j}$ is cancelled by $\frac{1}{c}\partial_t\vec{\nabla}\phi$

It is always possible to satisfy the Coulomb gauge $\quad \vec{\nabla}\cdot\vec{A}'=\vec{\nabla}\cdot\vec{A}+\vec{\nabla}^2\chi=0$

just solve for χ

Lorentz gauge $\qquad \vec{\nabla}\cdot\vec{A}+\frac{1}{c}\partial_t\phi=0$

$$\Rightarrow \quad -\vec{\nabla}^2 A+\frac{1}{c^2}\partial_t^2 A=\frac{4\pi}{c}j$$

$$-\vec{\nabla}^2\phi+\frac{1}{c^2}\partial_t^2\phi=4\pi\rho$$

We have no instantenous Coulomb potential because of the ∂_t^2 derivative

Can we always satisfy the Lorentz gauge

$$\vec{\nabla} A' + \frac{1}{c}\partial_t \phi' = \vec{\nabla}\vec{A} + \vec{\nabla}^2 \chi + \frac{1}{c}\partial_t \phi - \frac{1}{c^2}\partial_t^2 \chi$$
$$= 0$$

$$\Rightarrow \vec{\nabla}\cdot\vec{A} + \frac{1}{c}\partial_t \phi = -\vec{\nabla}^2 \chi + \frac{1}{c^2}\partial_t^2 \chi$$

can be solved for χ

still $\exists$ residual gauge invariant

for functions that satisfy

$$-\vec{\nabla}^2 \chi + \frac{1}{c^2}\partial_t \chi = 0$$

$$\Box = \vec{\nabla}^2 - \frac{1}{c^2}\partial_t^2$$

Green's functions

$$\Box \psi = -4\pi f \quad (**)$$

Green function $\quad \Box G = -4\pi \delta^3(\vec{r}-\vec{r}') \times \delta(t-t')$

$$G = G(\vec{r}, \vec{r}', t, t')$$

Then the solution of $(**)$ is given by

$$\psi = \int d^3 r' dt' \, G(\vec{r}, \vec{r}', t, t') f(\vec{r}', t) + \chi$$

with $\Box \chi = 0$

χ can be fixed by boundary conditions.

We look for a translational invariant solution. So we can put one of the arguments to 0 $\vec{r}' \to 0$, $t' \to 0$

$$\Rightarrow \Box G = -4\pi\, \delta^3(\vec{r})\, \delta(t)$$

Fourier transform

$$G(r,t) = \int d^3k\, d\omega\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}\, G(\vec{k},\omega)$$

$$\delta^3(\vec{r})\,\delta(t) = \int \frac{d^3k\, d\omega}{(2\pi)^4}\, e^{i(\vec{k}\vec{r}-\omega t)}$$

$$\Rightarrow \left(-k^2 + \frac{\omega^2}{c^2}\right) G(k,\omega) = \frac{-4\pi}{(2\pi)^4}$$

$$\Rightarrow G(\vec{k},\omega) = -\frac{1}{4\pi^3}\, \frac{1}{\frac{\omega^2}{c^2} - k^2}$$

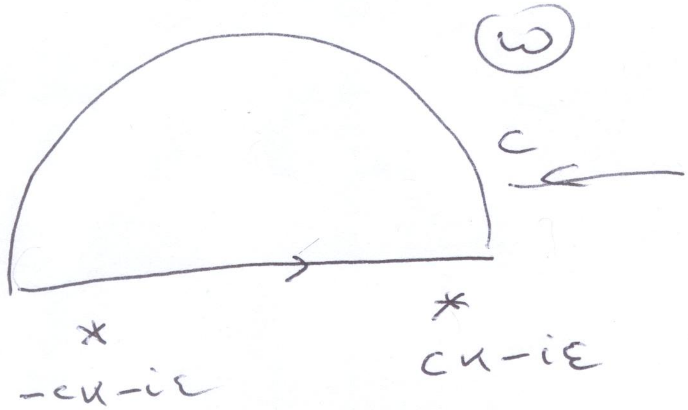
We implement the boundary condition by an $i\varepsilon$ presription $\omega \to \omega \pm i\varepsilon$

<u>Retarded and advanced Green's function</u>

$$G(\vec{r},t) = \int d^3k\, e^{ikr} \int_{-\infty}^{\infty} \frac{e^{-i\omega t}}{\frac{\omega^2}{c^2} - k^2}\, \frac{(-1)}{(4\pi)^3}$$

$$\frac{\omega^2}{c^2} - k^2 \longrightarrow \frac{(\omega + i\varepsilon)^2}{c^2} - k^2 \Rightarrow \text{poles}$$

are at $\omega = \pm ck - i\varepsilon$



$\omega + i\varepsilon$

$e^{-i\omega t} \to 0$ for $t<0$

$$\Rightarrow \int_{-\infty}^{\infty} d\omega \dots = \int_C d\omega \dots = 0$$

$$\Rightarrow \text{for } t<0 \quad \int_{-\infty}^{\infty} \frac{e^{-i\omega t}}{\frac{\omega^2}{\kappa^2} - k^2} = 0$$

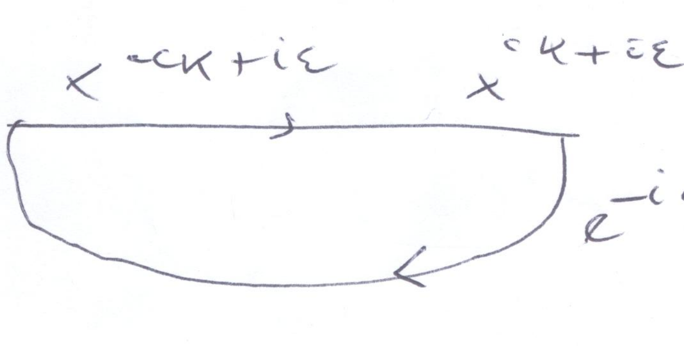
$$\Rightarrow G(\vec{r},t) = 0 \quad \text{for } t<0$$

This is the retarded Green's function

The advanced Green's function is defined by

$$G(\vec{r},t) = 0 \quad \text{for } t>0$$

This is achieved by $\omega \to \omega - i\varepsilon$ so that there are no poles in the lower half plane



$e^{-i\omega t} \to 0$ for $t>0$

Lecture #25

$$\vec\nabla^2 \vec A - \frac{1}{c^2}\partial_t^2 A = -\frac{4\pi}{c}\vec J$$

$$\vec\nabla^2 \phi - \frac{1}{c^2}\partial_t^2 \phi = -4\pi\rho$$

$$\Box = \vec\nabla^2 - \frac{1}{c^2}\partial_t^2$$

Green $\Box\, G(\vec r,\vec r\,',t,t') = 4\pi\,\delta^3(\vec r-\vec r\,') \times \delta(t-t')$

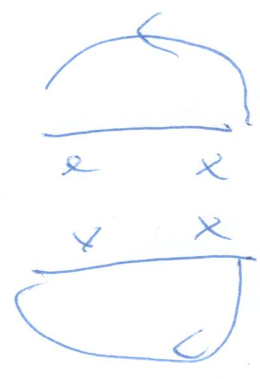
$$\Box\,\psi = -4\pi f$$

$$\Rightarrow \psi = \int d^3r'\,dt'\,G(\vec r,\vec r\,',t,t')\,f(\vec r\,',t') + \chi$$

with $\Box\,\chi = 0$

$$G(k,\omega) = -\frac{1}{4\pi^3}\,\frac{1}{\frac{\omega^2}{c^2}-k^2}$$

$\omega \to \omega \pm i\varepsilon$



retarded

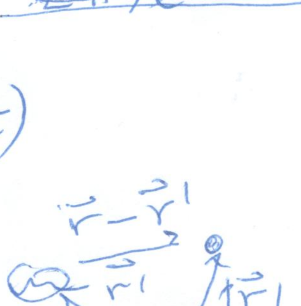
advanced.

Quiz $\int \frac{e^{ix}}{x+i}$

$2\pi i$, $-2\pi i$, 1, -1, $-2\pi i e$, $2\pi i e$

$2\pi e$ $-2\pi e$, ~~$2\pi/e$ $-2\pi/e$~~ 0

Today $G(\vec r,t)$



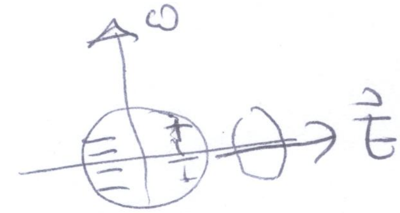
observation at $\vec r,t$

emission at $t-\frac{|\vec r-\vec r\,'|}{c}$

Lecture #3.

cauchy's theorem

Today $G(\vec r, t, \vec r', t')$



Magnetized sphere

$$\Phi_M = -\int_V \frac{\vec\nabla\cdot M(x')}{|x-x'|}\, d^3x' + \oint \frac{n'\cdot M(x')}{|x-x'|}\, ds$$

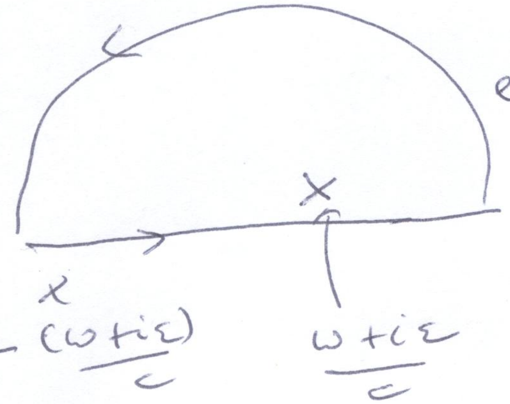
(5.100) of Jackson

$$\vec M = M\hat z \Rightarrow \vec\nabla\cdot M = 0$$

Calculation of $G(\vec{r},t)$

$$G_R(\vec{r},t) = \int d\omega\, e^{-i\omega t}\,\frac{1}{4\pi^3}\int 2\pi k^2\, dk \sin\theta\, d\theta \;\frac{e^{ikr\cos\theta}}{k^2 - \frac{(\omega+i\varepsilon)^2}{c^2}}$$

k_2



$e^{ikr} \to 0$

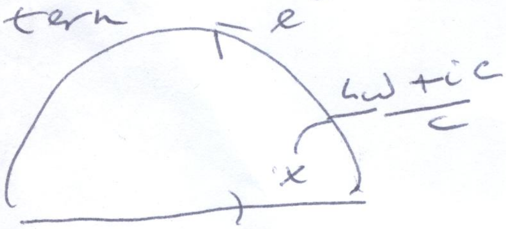
- First do θ integral $\quad \cos\theta = x$

$$\int_0^\pi \sin\theta\, d\theta\; e^{ikr\cos\theta} = \frac{1}{ikr}\left(e^{ikr} - e^{-ikr}\right)$$

$$\int_0^\infty \frac{k^2\, dk\,\left(e^{ikr} - e^{-ikr}\right)}{\left(k^2 - \frac{(\omega+i\varepsilon)^2}{c^2}\right) ikr} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{k^2\, dk\,\left(e^{ikr} - e^{-ikr}\right)}{\left(k^2 - \frac{(\omega+i\varepsilon)^2}{c^2}\right) ikr}$$

$$\frac{1}{\left(k - \frac{\omega+i\varepsilon}{c}\right)\left(k + \frac{\omega+i\varepsilon}{c}\right)}$$

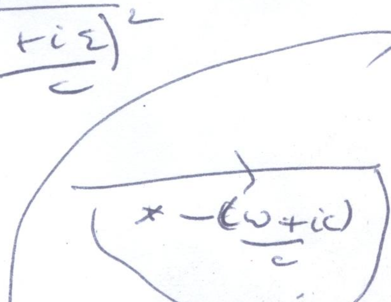
e^{ikr} term $\quad e^{ikr} \to 0$



$$\frac{1}{2}\, 2\pi i\, \frac{\left(\frac{\omega+i\varepsilon}{c}\right)^2}{2\left(\frac{\omega+i\varepsilon}{c}\right)^2}\, e^{i\left(\frac{\omega+i\varepsilon}{c}\right) r}$$

$$-\frac{1}{2}\,\frac{2\pi i \left(\frac{\omega+i\varepsilon}{c}\right)^2 (-)\, e^{ik\left(\frac{\omega+i\varepsilon}{c}\right) r}}{-2\frac{(\omega+i\varepsilon)}{c}\left(-\frac{(\omega+i\varepsilon)}{c}\right)}$$

e^{-ikr} term



$e^{-ikr} \to 0$

− sign for clockwise contour

The two integral are the same.

$$G_R(\vec{r},t) = \int d\omega\, e^{-i\omega t} \frac{2\pi}{4\pi^3} \frac{2\pi i 2}{4} e^{i\left(\frac{\omega \mp i\epsilon}{c}\right) r}$$

$$= \frac{1}{2\pi r} \int d\omega \; e^{-i\omega t + i\frac{\omega r}{c}}$$

$$= \frac{1}{2\pi r} 2\pi\, \delta\left(\frac{r}{c} - t\right)$$

$$= \frac{c}{r}\, \delta(r - ct)\, \Theta(t)$$

$\uparrow$ because $r > 0$

For $G_A(r,t)$ the analogous calculation gives

$$G_A(\vec{r},t) = \frac{c}{r}\, \delta(r + ct)\, \Theta(-t)$$

Therefore the solution of the PDE is given by

$$\psi(r,t) = \int d^3r'\, dt' \frac{\delta\left(t - t' - \frac{|\vec{r} - \vec{r}'|}{c}\right)}{|r - r'|} F(\vec{r}', t)$$

retarded time $\quad t_R = t - \frac{|\vec{r} - \vec{r}'|}{c}$

this is the time at which a signal must be emitted to be observed at $(\vec{r},t)$

Solution for $\vec{A}$ and ϕ

$$\vec{A} = \int \frac{d^3r'}{|\vec{r} - \vec{r}'|}\, \vec{J}(\vec{r}', t_R)$$

no instanteous fields!

$$\phi = \int \frac{d^3r'}{|\vec{r} - r'|}\, \rho(r', t_R)$$

Lecture #31

$$G_R(\vec{r},t) = \frac{c}{r}\,\delta(r - ct)\,\theta(t)$$

$$t_R = t - \frac{|\vec{r} - \vec{r}'|}{c}$$

$$\vec{A} = \int \frac{d^3r'}{|\vec{r}-\vec{r}'|}\,\frac{\vec{J}}{c}(\vec{r}', t_R)$$

$$\phi = \int \frac{d^3r'}{|\vec{r}-\vec{r}'|}\,\rho(\vec{r}', t_R)$$

Today: radiation of a harmonic source

Radiation

$$\vec{j}(\vec{r},t) = \vec{j}_0(\vec{r})\, e^{-i\omega t}$$

$$\rho(\vec{r},t) = \rho_0(\vec{r})\, e^{-i\omega t}$$

source

Continuity eq. $\vec{\nabla}\, \vec{j}_0 = i\omega\rho_0$

Far away, the energy density per unit solid angle is constant. This is the radiation zone.

$$d\vec{s} \cdot \vec{P} = \frac{c}{4\pi}\, \vec{E} \times \vec{B} \cdot d\vec{s}$$

$d\vec{s}$

$$= \frac{c}{4\pi} (\vec{E} \times \vec{B}) \cdot \hat{r}\, r^2 d\Omega$$

this can only be constant if $\vec{E} \sim \frac{1}{r}$, $\vec{B} \sim \frac{1}{r}$

$$\vec{A} = \frac{1}{c} \int \frac{d^3 r'}{|r - r'|}\, \vec{j}_0(\vec{r}')\, e^{-i\omega t_R}$$

$$\phi = \int \frac{d^3 r'}{|r - r'|}\, \rho_0(\vec{r}')\, e^{-i\omega t_R}$$

We only need leading order terms in $\frac{1}{r}$

$$t_R = t - \frac{|r - r'|}{c} = t - \frac{1}{c}\sqrt{r'^2 + r'^2 - 2\vec{r}\cdot\vec{r}'}$$

$$= t - \frac{r}{c} - \frac{\vec{r}\cdot\vec{r}'}{c} + O\left(\frac{r'^2}{r^2}\frac{r}{c}\right)$$

$$\frac{1}{|r - r'|} = \frac{1}{r} + O\left(\frac{r'}{r^2}\right)$$

For $r \gg r'$ we can neglect the corrections

$$\Rightarrow A(\vec{r},t) = \frac{e^{-i\omega(t-\frac{r}{c})}}{rc} \int d^3r' \, e^{-i\omega\frac{\vec{r}\cdot\vec{r}'}{cr}} \vec{j}_0(\vec{r}')$$

$$\phi(\vec{r},t) = \frac{e^{-i\omega(t-\frac{r}{c})}}{r} \int d^3r' \, e^{-i\omega\frac{\vec{r}\cdot\vec{r}'}{cr}} \rho_0(\vec{r}')$$

Physical fields

$$\vec{E} = -\frac{1}{c}\partial_t \vec{A} \cancel{+} - \vec{\nabla}\phi$$

$$= \frac{e^{-i\omega(t-\frac{r}{c})}}{r} \int d^3r' \, e^{-i\omega\frac{(\vec{r}\cdot\vec{r}'}{cr}} \times \left(\frac{i\omega}{c^2}\vec{j}_0 - \frac{i\omega}{c}\frac{\vec{r}}{r}\rho_0 \right) + O\left(\frac{1}{r^2}\right)$$

$$\vec{B} = \vec{\nabla}\times\vec{A} = \frac{i\omega}{c^2}\frac{\vec{r}}{r^2} \times e^{-i\omega(t-\frac{r}{c})} \times \int d^3r' \, e^{-i\omega\frac{\vec{r}\cdot\vec{r}'}{cr}} \vec{j}_0(r')$$

~~taken~~ inner product with $\hat{r}$

For a wave with $\vec{k} = \frac{\omega}{c}\frac{\vec{r}}{r}$ we have

$\vec{E} \perp k$:

continuity eq.

$$\int d^3r' \, \frac{i\omega}{c}\rho_0 \, e^{-i\omega\frac{\vec{r}\cdot\vec{r}'}{cr}} = \frac{1}{c}\int d^3r' \, \nabla'_j \vec{j}_0(r') \, e^{-i\omega\frac{\vec{r}\cdot\vec{r}'}{cr}}$$

partial integr.

$$= -\int d^3r \, \vec{j}_0(\vec{r}') \left(-\frac{i\omega}{c^2}\frac{\vec{r}}{r}\right) e^{-i\omega\frac{\vec{r}\cdot\vec{r}'}{cr}}$$

$$\Rightarrow \vec{E}\cdot\frac{\vec{r}}{r} = 0 \rightarrow \vec{E}\cdot\vec{k} = 0$$

– Longitudinal part of $\vec{j}_0$ is cancelled by $\frac{c\vec{r}}{r}\rho_0$. ✓

Longitudinal part of $\vec{j}_0$ does not contribute to $\vec{B}$

$\hat{r}\times\hat{r}=0$

Because $-\frac{i\omega}{c}\frac{\vec{r}}{r}\rho_0$ and the longitudinal part of $\vec{j}_0$ is cancelled we have

$$\vec{B}=\hat{k}\times\vec{E}$$

Radiated energy

$$\frac{dW}{dt\,dA}=\vec{P}\cdot\hat{r}\Rightarrow\frac{dW}{dt\,d\Omega}=r^2\,\vec{P}\cdot\hat{r}$$

$$=r^2\frac{c}{8\pi}\mathrm{Re}\,\vec{E}^*\times\underbrace{\vec{B}}_{\hat{k}\times\vec{E}}\cdot\hat{r}$$

$$\vec{E}^*\times(\hat{k}\times\vec{E})=\hat{k}\,\vec{E}^*\cdot E$$

$$\Rightarrow\frac{dW}{dt\,d\Omega}=\frac{cr^2}{8\pi}\,\vec{E}^*\cdot E$$

$$=\frac{cr^2}{8\pi}\frac{1}{r^2}\left|\int d^3r'\,e^{-i\omega\frac{\vec{r}\cdot\vec{r}'}{cr}}\,j_{0T}\frac{i\omega}{c^2}\right|^2$$

$$\vec{j}_T=\vec{j}-c\frac{\vec{r}}{r}\rho$$

$$\Rightarrow\frac{dW}{dt\,d\Omega}=\frac{c\omega^2}{c^3\,8\pi}\left|\int d^3r'\,e^{-i\vec{k}r'}\,j_T(r')\right|^2$$

Lecture #32

$$\vec{j} = \vec{j}_0 e^{-i\omega t} \qquad \rho = \rho_0 e^{-i\omega t}$$

9.1a

Longitudinal component of j_0 is cancelled by ρ_0 term.

$$\vec{E} = \frac{e^{-i\omega(t - \frac{r}{c})}}{r} \int d^3r\, e^{-i\omega(\hat{r}\cdot\vec{r}')/cr} \frac{i\omega}{c} \vec{j}_0^{\,T}$$

$$\vec{B} = \vec{k} \times \vec{E} \qquad \vec{k} = |k|\,\hat{r}$$

$$\frac{dw}{dt\,d\Omega} = \frac{c\,\omega^2}{c^2\, 8\pi} \left| \int d^3r'\, e^{-i\hat{k}\cdot\vec{r}'} \vec{j}^{\,T}(r') \right|$$

Today electric dipole radiation
magnetic dipole radiation.

Electric dipole radiation

(gp)

harmonic source

$$\vec{A} = \frac{e^{-i\omega(t-\frac{r}{c})}}{cr} \int d^3r' \, e^{-i\frac{\omega}{c}\hat{r}.\vec{r}'} \vec{j}_0(r')$$

$$\phi = \frac{e^{-i\omega(t-\frac{r}{c})}}{r} \int d^3r' \, e^{-i\frac{\omega}{c}\hat{r}.r'} \rho(r')$$

$$\omega = kc = \frac{2\pi c}{\lambda} \Rightarrow \frac{\omega}{c} = \frac{2\pi}{\lambda}$$

wavelength much larger than the size of the source then $|\frac{\omega}{c}\hat{r}.\vec{r}'| \ll 1$

$$e^{-i\frac{\omega}{c}\hat{r}.r'} = 1 - \frac{i\omega}{c}\hat{r}.r'$$

to leading order we only keep this term

$$\int d^3r' \, j_1(r') = -\int d^3r' \, x_1 \partial_1 j_1(r')$$

one component

$$= -\int d^3r' \left(-\partial_2 j_2 - \partial_3 j_3 - \partial_t \rho\right) x_1$$

$$\int d^3r' \, \partial_2 j_2 = 0$$

$$\int d^3r' \, \partial_3 j_3 = 0$$

$$= \int d^3r' \, \partial_t \rho \, x_1 = \partial_t \int d^3r' \, \rho \, x_1$$

$$= \partial_t P_1$$

one component of dipole moment

$$\Rightarrow \vec{A} = \frac{e^{-i\omega(t-\frac{r}{c})}}{cr}(-i\omega)\vec{P}$$

$$\vec{B} = \vec{\nabla}\times\vec{A} = \frac{i\omega}{c^2}\hat{r}\times p \, e^{-i\omega(t-\frac{r}{c})} \frac{(-i\omega)}{r}$$

$$\vec{B} = \frac{\omega^2}{rc^2}\hat{r}\times\vec{p}\; e^{-i\omega(t-\frac{r}{c})}$$

away from the sources we have $\vec{j} = 0$

$$\Rightarrow \partial_t E = c\vec{\nabla}\times\vec{B} - \underbrace{4\pi\vec{j}}_{0}$$

$$\Rightarrow E = \frac{ic}{\omega}\vec{\nabla}\times\vec{B}$$

Far away from the source (radiation zone) we have

$$\vec{E} = \frac{ic}{\omega}\frac{i\omega}{c}\hat{r}\times(\hat{r}\times\vec{p})\, e^{-i\omega(t-\frac{r}{c})}\frac{\omega^2}{rc^2}$$

$$= -\frac{\omega^2}{rc^2}\left((\hat{r}\cdot\vec{p})\hat{r} - \vec{p}\right) e^{-i\omega(t-\frac{r}{c})}$$

$$\Rightarrow \vec{B} = \hat{r}\times\vec{E}$$

$$\vec{E}\cdot\hat{r} = 0$$

Poynting vector $\vec{P} = \frac{c}{8\pi}\mathrm{Re}(\vec{E}\times\vec{B}^*)$

$$= \frac{c}{8\pi}\mathrm{Re}(\vec{E}\times(\hat{r}\times\vec{E}^*))$$

$$= \frac{c}{8\pi}\hat{r}|\vec{E}|^2$$

$$\frac{dW}{dt\,d\Omega} = \vec{P}\cdot\hat{r}\, r^2 = \frac{c}{8\pi}\frac{r^2\omega^4}{r^2c^{3\cancel{4}}}\left(\vec{p}^2 - (\hat{r}\cdot\vec{p})^2\right)$$

$$= \frac{\cancel{c}1}{8\pi}\frac{\omega^4}{c^{3\cancel{4}}}\vec{p}^2\sin^2\theta$$

ecture # 33

$$\vec{A}(\vec{r},t) = \frac{e^{-i\omega(t-\frac{r}{c})}}{rc} \int d^3r' \, e^{-i\frac{\omega}{c}\hat{r}\cdot\vec{r}'} \, \vec{j}_0(r')$$

$\vec{j}(\vec{r}',t)$

$$\phi(\vec{r},t) = \frac{e^{-i\omega(t-\frac{r}{c})}}{r} \int d^3r' \, e^{-i\frac{\omega}{c}\hat{r}\cdot\vec{r}'} \, \rho_0(\vec{r}')$$

$\Rightarrow \rho(\vec{r}',t)$

$$e^{-i\frac{\omega}{c}\hat{r}\cdot\vec{r}'} = 1 - i\frac{\omega}{c}\hat{r}\cdot\vec{r}' + \dots$$

$\uparrow$ electric dipole radiation

$\nwarrow$ magnetic dipole and electric quadrupod

Today – magnetic dipole
– electric quadrupole
– radiation of moving charge

$$\frac{dW}{dt\,d\Omega} = \frac{c}{8\pi}\frac{\omega^4}{c^3}\vec{p}^2\sin^2\theta$$

Magnetic dipole radiation

We use the next term in the expansion

$$e^{-i\vec{k}.\vec{r}'} = 1 - i\vec{k}.\vec{r}$$

$$\int d^3r'\, e^{-i\vec{k}.\vec{r}'}\, j_0(r') \to$$

$$\int d^3r'\, (-i\vec{k}.\vec{r}')\, j_0(r')$$

$$-ik_j r'_j j_{0\ell} = -ik_j\left((x'_j j_{0\ell} - x'_\ell j_{0j})/2\right.$$

$$\left. - (x'_j j_{0\ell} + x'_\ell j_{0j})/2\right)$$

This gives

$$\int d^3r\, (-i\vec{k}.r')\, j_0(r') = \int d^3r'\, (i\vec{k} \times (\vec{r}' \times \frac{\vec{j}_0(r')}{2}))_\ell$$

$$+ \frac{ik_i}{2} \int d^3r'\, x'_i x'_\ell\, \partial'_k \cdot j_{0k}$$

Partial integrate

go back

$-\partial_t \rho$

$$= i\vec{k}c \times \vec{p}_M \qquad - \frac{\omega k_i}{2} \int d^3r'\, x'_i x'_\ell\, \rho_0(\vec{r}')$$

magnetic dipole moment

this givs quadrupole radiation

$$\vec{p}_M = \int d^3r'\, \frac{\vec{r}' \times \vec{j}_0}{2c}$$

Magnetic dipole radiation

(101)

$$\vec{A} = \frac{e^{-i\omega(t-r/c)}}{cr} \, c\, i\vec{k} \times \vec{P}_M$$

$$\vec{B}_{M_1} = \vec{\nabla} \times \vec{A} = i\vec{k} \times A + O\left(\frac{1}{r^2}\right)$$

$$= -\vec{k} \times (\vec{k} \times \vec{P}_M) \frac{e^{-i\omega(t-\frac{r}{c})}}{r}$$

$$E_{M_1} = \frac{i}{k} \vec{\nabla} \times \vec{B}_{M_1} = \frac{i}{k}\, i\vec{k} \times \vec{B}_{M_1} + O\left(\frac{1}{r^2}\right)$$

$$\Big(\partial_t \vec{E} = c\vec{\nabla} \times \vec{B} - \underset{0}{\underbrace{4\pi \vec{j}}}\Big) = -\frac{\vec{k}}{k} \times \vec{B}_M = -\hat{k} \times \vec{B}_{M_1}$$

$$\Rightarrow \frac{dW}{dt\, d\Omega} = \frac{\omega^4}{8\pi c^3} P_M^2 \sin^2\theta$$

Electric quadrupole radiation

$$\vec{A}_{E_2} = \frac{e^{-i\omega(t-\frac{r}{c})}}{rc} \int d^3r' \, -\frac{\omega k_i}{2} x_i' x_l' \rho_0(r')$$

$$\vec{B} = \vec{\nabla} \times \vec{A} = i\vec{k} \times \vec{A} + O\left(\frac{1}{r^2}\right)$$

$$\vec{B}_r = \frac{i\, e^{-i\omega(t-\frac{r}{c})}}{rc} \varepsilon_{pqr} k_q \frac{\omega k_i}{2} \overbrace{\int d^3r' \left(x_i' x_r' - \frac{1}{3}\delta_{ir}\right) \times \rho_0(r', t)}^{2Q_{ir}}$$

vanishes when contracted with ε

$$\vec{B}_{E_2} = \frac{e^{-i\omega(t-r/c)}}{rc} \varepsilon_{pqr} k_q Q_{ik} k_i$$

Q_{ir} is the quadrupole moment

$$\vec{E}_{E_2} = \frac{i}{k}\, i\vec{k} \times \vec{B}_{E_2}$$

Lecture #34 (10cw)

E_1, E_2, M_1 radiation

$$E_1: \quad \frac{c}{8\pi}\frac{\omega^4}{c^4}\,\vec{p}^{\,2}\sin^2\theta = \frac{dW}{dt\,d\Omega}$$

$$M_1 \quad \frac{\omega^4}{8\pi c^3}\,\vec{p}_{\Omega}^{\,2}\sin^2\theta = \frac{dW}{dt\,d\Omega}$$

<u>Today</u> Radiation of moving charge
Larmor formula

Radiation by slowly moving charges

$$\vec{j} = q\,\vec{v}(t)\,\delta(\vec{r} - r_p(t))$$

$$\vec{A} = \frac{1}{c}\int d\vec{r}'\,dt'\,c\,\frac{\delta(|r-r'| - (t-t')c)}{|r-r'|}\,\vec{j}(\vec{r}',t')$$

$$= \frac{1}{c}\int dt'\,q\,\vec{v}(t')\cdot\frac{\delta(|\vec{r}-\vec{r}_p(t')| - (t-t')c)}{|\vec{r}-\vec{r}_p|}$$

$$\delta(f(x)) = \frac{1}{f'(x)}\,\delta(x)$$

$$\partial_t\,[\,|\vec{r}-\vec{r}_p(t')| - (t-t')c\,]$$

$$= c - \frac{(\vec{r}-\vec{r}_p)\cdot\dot{r}_p}{|\vec{r}-r_p|}$$

For $r \to \infty$ $\quad \simeq c - \hat{r}\cdot\dot{\vec{r}}_p$

$$\Rightarrow A = \frac{q}{c}\,\frac{v(t_R)}{r}\,\frac{1}{1-\frac{\hat{r}\cdot\dot{r}_p}{c}(t_R)} + O\left(\frac{1}{r^2}\right)$$

$$\phi = \int d^3r'\,dt'\,\frac{\delta(|r-r'| - (t-t')c)}{|r-r'|}\,\rho(r',t')$$

$$\rho' = q\,\delta(\vec{r}-\vec{r}_p(t))$$

$$\Rightarrow \phi = \frac{q}{r}\,\frac{1}{1-\frac{\hat{r}\cdot\dot{\vec{r}}_p(t_R)}{c}} + O\left(\frac{1}{r^2}\right)$$

$$t_R \simeq t - \frac{r}{c}$$

$$\vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\partial_t \vec{A}$$

$$= -\frac{q}{r}\frac{-1}{\left(1 - \frac{\hat{r}\cdot\vec{r}_p}{c}\right)}c\left(\frac{\ddot{\vec{r}}_p\cdot\hat{r}}{c^2}\right)\hat{r}$$

$$-\frac{q}{c}\frac{\dot{v}(t_R)}{c} + \mathcal{O}\left(\frac{v}{c}\right)$$

$$-\vec{\nabla}(\dot{\vec{r}}_p(t_R)) = \vec{\nabla}\left(\dot{r}_p\left(t - \frac{r}{c}\right)\right)$$

$$= -\ddot{\vec{r}}_p(t_R)\cdot\hat{r}\left(-\frac{1}{c}\right)\frac{\vec{r}}{r}$$

$$\vec{E} = -\frac{q}{c^2 r}\left(\vec{a} - (\vec{a}\cdot\hat{r})\hat{r}\right) + \mathcal{O}\left(\frac{v}{c}\right) + \mathcal{O}\left(\frac{1}{r^2}\right)$$

$$\vec{B} = (\vec{\nabla}\times\vec{A})_i = \frac{q}{c}\vec{\nabla}\times\left(\frac{v_p\left(t-\frac{r}{c}\right)}{r}\frac{1}{1-\frac{\hat{r}\,\dot{r}_p(t_R)}{c}}\right)$$

$$\uparrow$$
$$\mathcal{O}\left(\frac{v}{c}\right)$$

$$= \frac{q}{c}\varepsilon_{ijk}\partial_j v_{p,k}\left(t - \frac{r}{c}\right)$$

$$= \frac{q}{c}\dot{v}_{p,k}\,\frac{-1}{c}\hat{r}_j\,\varepsilon_{ijk}$$

$$= -\frac{q}{c^2}\hat{r}\times\dot{\vec{v}}_p \Rightarrow \vec{B} = \hat{r}\times\vec{E}$$

note that $\hat{r}\times\hat{r} = 0$

$$\vec{E}\cdot\hat{r} = -\frac{q}{c^2 r}(\vec{a}\cdot\hat{r} - \vec{a}\cdot\hat{r}) = 0$$

$\Rightarrow$ we have a transverse wave

Poynting vector

$$\vec{P} = \frac{c}{4\pi}(\vec{E}\times\vec{B}) = \frac{c}{4\pi}(\vec{E}\times(\hat{r}\times\vec{E}))$$

$$= \frac{c}{4\pi}\vec{E}^2\hat{r}$$

$$= \frac{c}{4\pi}\frac{q^2}{c^4 r^2}\left|\vec{a} - (\vec{a}\cdot\hat{r})\hat{r}\right|^2\hat{r}$$

radiation per unit of solid angle.

$$\hat{r}\cdot d\vec{P} = r^2\, d\Omega\, \vec{P}\cdot\hat{r}$$

$$= r^2\frac{c}{4\pi}\frac{q^2}{c^4 r^2}\left|\vec{a}^2 - \underbrace{(a\cdot\hat{r})}_{=\,a\cos\theta}{}^2\right| d\Omega$$

$$\Rightarrow \frac{dP}{d\Omega} = \frac{q^2 a^2}{4\pi c^3}\sin^2\theta$$

Total power $P = \int d\Omega\, dP/d\Omega$

$$= \int d\varphi \sin\theta\, d\theta\, \frac{q^2 a^2}{4\pi c^3}\sin^2\theta$$

$$= 2\pi\frac{q^2 a^2}{4\pi c^3}\int \sin^3\theta\, d\theta$$

$$= \frac{2}{3}\frac{q^2 a^2}{c^3}$$ Larmor formula

<u>Example</u> radiating point particle

$$r(t) = \mathrm{Re}\,(\vec{r}_0\, e^{-i\omega t}) \Rightarrow \vec{p} = q\,\vec{r}_0$$

$$\vec{a}(t) = \mathrm{Re}\,(r_0\,(-i\omega)^2 e^{-i\omega t})$$

$$\frac{dW}{d\Omega\, dt} = \frac{q^2}{4\pi c^3}\,\overline{\vec{a}^2}\,\sin^2\theta$$

$$= \frac{q^2}{4\pi c^3}\, p^2 \omega^4 \sin^2\theta\, \overline{\cos^2 \omega t}$$

$$\overline{\cos^2\omega t} \downarrow \frac{1}{2}$$

$$= \frac{\omega^4}{8\pi c^3}\, \vec{p}^{\,2} \sin^2\theta$$

which is exactly the result from the dipole approximation.