

Time dependent ME in vacuum

$\rho = 0 \quad \vec{j} = 0 \Rightarrow \text{div } \vec{B} = 0 \quad \text{div } \vec{E} = 0$

$\vec{\nabla} \times \vec{E} = -\frac{1}{c} \partial_t \vec{B} \Rightarrow \vec{\nabla} \times \partial_t \vec{E} = -\frac{1}{c} \partial_t^2 \vec{E}$

$\vec{\nabla} \times \vec{B} = \frac{1}{c} \partial_t \vec{E} \Rightarrow \vec{\nabla} \times \partial_t \vec{B} = \frac{1}{c} \partial_t^2 \vec{E}$

$\Rightarrow c \vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = -\frac{1}{c} \partial_t^2 \vec{B}$   
 $-c \vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \frac{1}{c} \partial_t^2 \vec{E}$

$(\vec{\nabla} \times (\vec{\nabla} \times B))_i = \epsilon_{ijk} \epsilon_{kpq} \partial_j \partial_p B_q$

$= (\delta_{ip} \delta_{jq} - \delta_{iq} \delta_{jp}) \partial_j \partial_p B_q$

$= \partial_j \partial_i B_j - \partial_j^2 B_i$

$\underbrace{\partial_i (\partial_j B_j)}_0$

$\Rightarrow \vec{\nabla}^2 \vec{B} = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \vec{B} = 0$

same  $\vec{\nabla}^2 \vec{E} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \vec{E} = 0$

These are wave equations.

Each component satisfies

$(\partial_x^2 + \partial_y^2 + \partial_z^2) F - \frac{1}{c^2} \partial_t^2 F = 0$

Let us look at 1d.

$\partial_x^2 F - \frac{1}{c^2} \partial_t^2 F = 0$

Normally we expand in plane waves

$$\sum_{\vec{k}} a_{\vec{k}} e^{i(\vec{k} \cdot \vec{x} - \omega t)}$$

Since the equations are linear we can solve them component by component

$$\Rightarrow \vec{E}_{\vec{k}} = \text{Re} \vec{e}_{\vec{k}} e^{i(\vec{k} \cdot \vec{x} \pm \omega t)}$$

↑  
polarization of the wave

$$\vec{\nabla} \times \vec{B} = \frac{1}{c} \partial_t \vec{E}$$

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c} \partial_t \vec{B}$$

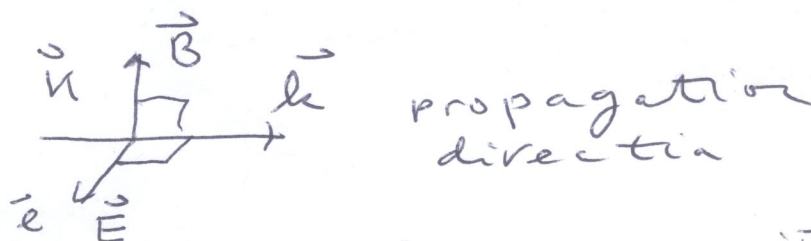
$$(i\vec{k} \times \vec{e}) e^{i(\vec{k} \cdot \vec{x} - \omega t)} = -\frac{1}{c} \partial_t \vec{B} e^{i(\vec{k} \cdot \vec{x} - \omega t)}$$

$$\Rightarrow \vec{B} = (i\vec{k} \times \vec{e}) e^{i\vec{k} \cdot \vec{x}} \frac{c}{i\omega} e^{-i\omega t}$$

physical component is real,

$$\Rightarrow \vec{B} = \text{Re} \left( \frac{c\vec{k}}{\omega} \times \vec{e} \right) e^{i\vec{k} \cdot \vec{x} - i\omega t}$$

wave



$$\frac{1}{c} \partial_t \vec{E} = \vec{\nabla} \times \vec{B} = \frac{i\vec{k} \times (c\vec{e} \times \vec{e})}{i\omega} e^{i\vec{k} \cdot \vec{x} - i\omega t}$$

Why is  $\vec{E} \perp \vec{k}$  ?

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$$\vec{\nabla} \cdot \vec{E} = 0 \Rightarrow \vec{k} \cdot \vec{E} e^{i(\vec{k} \cdot \vec{r} - \omega t)} = 0$$
$$\Rightarrow \vec{k} \cdot \vec{E} = 0$$

Same for  $\vec{B}$ ,  $\vec{\nabla} \cdot \vec{B} = 0 \Rightarrow \vec{B} \perp \vec{k}$

polarization direction of  $\vec{E}$

$\vec{E} \perp \vec{k} \Rightarrow \exists$  two independent directions

- elliptical polarization  $\vec{E} = \vec{E}_1 + i\vec{E}_2$

- linear polarization  $\vec{E}_1 \parallel \vec{E}_2$

- circular polarization  $\vec{E}_1 \perp \vec{E}_2$

$\vec{E}_R = \vec{E}_x + i\vec{E}_y$  right handed polarized

$\vec{E}_L = \vec{E}_x - i\vec{E}_y$  left handed polarized

$$\text{Re } \vec{E} = \text{Re} (\vec{E}_x + i\vec{E}_y) e^{-i\omega t}$$

$$= \vec{E}_x \cos \omega t + \vec{E}_y \sin \omega t$$

this is the equation of a circle



# Time dependent Maxwell equations in matter in the dipole approximation (68)

approximation matter can be represented by a dipole density and a magnetic moment density

The dipole density gives  $\rho_{\text{bound}} = -\vec{\nabla}_r \cdot \vec{P}$

and the magnetic moment density

$$\vec{j}_b = c \vec{\nabla} \times \vec{M}$$

$$\Rightarrow 1) \vec{\nabla} \times \vec{E} = -\frac{1}{c} \partial_t \vec{B}$$

$$2) \vec{\nabla} \cdot \vec{B} = 0$$

$$3) \vec{\nabla} \cdot \vec{E} = 4\pi(\rho - \vec{\nabla} \cdot \vec{P})$$

$$4) \vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{j} + 4\pi \vec{\nabla} \times \vec{M}$$

(no need to include surface charge density and surface current because they can be represented as  $\delta$ -functions.)

However as in case of the vacuum eqs, they are not consistent

1) and 2) are consistent

$$0 = \vec{\nabla} \cdot \vec{\nabla} \times \vec{E} = -\frac{1}{c} \partial_t \vec{\nabla} \cdot \vec{B} = 0$$

$$\text{but } 0 = \vec{\nabla} \cdot \vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{\nabla} \cdot \vec{j} = -\frac{\partial \rho}{\partial t}$$

$$+ \frac{4\pi}{c} \vec{\nabla} \cdot \vec{j}_b + \vec{\nabla} \cdot \vec{\nabla} \times \vec{M} \stackrel{=0}{=} \text{add additional term}$$

From 3)  $\frac{\partial \rho}{\partial t} = \frac{1}{4\pi\epsilon_0} \nabla \cdot \vec{E} + \frac{1}{4\pi} \partial_t \nabla \cdot \vec{P}$

So we need

$$-\frac{1}{4\pi\epsilon_0} \frac{1}{4\pi} \nabla \cdot (\partial_t \vec{E} + \partial_t \vec{P}) + \frac{4\pi}{\epsilon} \nabla \cdot \vec{j}_b = 0$$

$$\Rightarrow \vec{j}_b = \frac{1}{4\pi} (\partial_t \vec{E} + \partial_t \vec{P}) + \nabla \times \vec{Q}$$

choose  $\vec{Q} = 0$

$$\Rightarrow \nabla \times \vec{B} = \frac{4\pi}{c} \vec{j} + \frac{1}{c} \partial_t \vec{E} + \frac{1}{c} \partial_t \vec{P} + 4\pi \vec{B} \times \vec{M}$$

$$\Rightarrow \nabla \times (\vec{B} - 4\pi \vec{M}) = \frac{4\pi}{c} \vec{j} + \frac{1}{c} \partial_t \vec{D}$$

$$\nabla \cdot (\vec{E} + \vec{P}) = 4\pi \rho$$

We have rewritten the  $\nabla \cdot \vec{E}$  in matter in terms of  $\vec{D}$  and  $\vec{H}$ :

$$\nabla \times \vec{E} = -\frac{1}{c} \partial_t \vec{B}$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \cdot \vec{D} = 4\pi \rho$$

$$\nabla \times \vec{H} = \frac{4\pi}{c} \vec{j} + \frac{1}{c} \partial_t \vec{D}$$

$\vec{P}$  is time dependent  $\Rightarrow \epsilon$  is time dependent



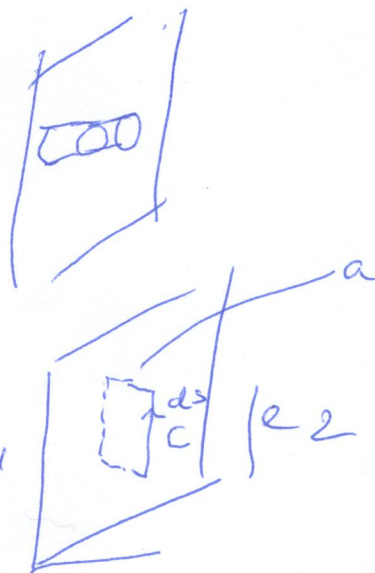
# Boundary conditions

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$$\vec{\nabla} \cdot \vec{B} = 0 \Rightarrow \Delta B_n = 0$$

$$\vec{\nabla} \cdot \vec{D} = 4\pi \rho \Rightarrow \Delta D_n = 4\pi \sigma$$

this can be proved  
using a Gaussian  
pillbox.



$$\vec{\nabla} \times \vec{E} = -\partial_t \vec{B}$$

$$\oint_C \vec{E} \cdot d\vec{s} = \int_S \vec{\nabla} \times \vec{E} \cdot \vec{n} \, d\vec{a}$$

$$= \int_S \partial_t \vec{B} \cdot \vec{n} \, d\vec{a}$$

$$= \partial_t \int_S \vec{B} \cdot \vec{n} \, d\vec{a}$$

$$\oint_C \vec{E} \cdot d\vec{s} = (E_{2 \tan} - E_{1 \tan}) l = \partial_t B_n a l$$

$\swarrow$  continuous  
 $a \rightarrow 0$

$$\Rightarrow E_{2 \tan} = E_{1 \tan}$$

For the  $\vec{\nabla} \times \vec{H}$  we obtain

$$\oint_C \vec{\nabla} \times \vec{H} \cdot d\vec{s} = \frac{4\pi}{c} \int_S \vec{j} \cdot d\vec{s} + \frac{1}{c} \int_S \partial_t \vec{D} \cdot d\vec{s}$$

$\frac{4\pi}{c} I$        $\frac{\partial \sigma a l}{\partial t a l}$

$$\oint_C \vec{H} \cdot d\vec{s}$$

$$= (H_{2 \tan} - H_{1 \tan}) l = \frac{4\pi}{c} K l$$

$a \rightarrow 0$

$$\Rightarrow \Delta H_{\text{em}} = \frac{4\pi}{c} \kappa$$

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Energy and momentum in matter

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c} \partial_t \vec{B}$$

$$\vec{\nabla} \times \vec{H} = \frac{1}{c} \partial_t \vec{D} + \frac{4\pi}{c} \vec{j}$$

Note that the equations in matter are an approximation contrary to the equations in vacuum.

then

$$\frac{1}{c} \vec{E} \partial_t \vec{D} + \frac{1}{c} \vec{H} \partial_t \vec{B} = -\frac{4\pi}{c} \vec{j} \cdot \vec{E} + \vec{E} \cdot \vec{\nabla} \times \vec{H} - \vec{H} \cdot \vec{\nabla} \times \vec{E}$$

$$\Rightarrow -\frac{4\pi}{c} \vec{j} \cdot \vec{E} - \vec{\nabla} \cdot (\vec{E} \times \vec{H})$$

$$\vec{\nabla} \cdot (\vec{E} \times \vec{H}) = \partial_i \epsilon_{ijk} E_j H_k$$

$$= \epsilon_{ijk} (\partial_i E_j) H_k + \epsilon_{ijk} E_j \partial_i H_k$$

$$= H_i \partial_i \vec{E} - \vec{E} \cdot \vec{\nabla} \times \vec{H}$$

integrate and  $-\frac{c}{4\pi}$

$$-\frac{1}{4\pi} \int d^3r \left( \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} + \vec{H} \cdot \frac{\partial \vec{B}}{\partial t} \right) = \int d^3r \vec{j} \cdot \vec{E}$$

$$+ \frac{c}{4\pi} \int d^3r \vec{\nabla} \cdot (\vec{E} \times \vec{H})$$



assum linear media (72)

$$\vec{D} = \epsilon \vec{E} \quad \vec{B} = \mu \vec{H}$$

$$\Rightarrow \frac{\partial}{\partial t} \left( -\frac{1}{8\pi} \int d^3r (\vec{E} \cdot \vec{D} + \vec{B} \cdot \vec{H}) \right)$$

extra factor  $\frac{1}{2}$

$$= \int d^3r \vec{j} \cdot \vec{E} + \frac{c}{4\pi} \int d^3r \vec{\nabla} \cdot \vec{E} \times \vec{H}$$

$$\frac{c}{4\pi} \int_{\partial V} d\vec{s} \cdot \vec{n} \cdot \vec{E} \times \vec{H}$$

decrease of energy of the medium

power transferred to current

energy flow out of the volume

Poynting vector  $\vec{P} = \frac{c}{4\pi} \vec{E} \times \vec{H}$

energy flow per unit area per second



## Free fields in isotropic media

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We want to derive a wave equation wh.

$$\epsilon_{ij}(\omega) = \delta_{ij} \epsilon(\omega)$$

$$\mu_{ij}(\omega) = \delta_{ij} \mu(\omega)$$

$$\sigma_{ij}(\omega) = \delta_{ij} \sigma(\omega)$$

$$\vec{j} = \sigma \vec{E}$$

$$\vec{b} = \epsilon \vec{E}$$

$$\vec{B} = \mu \vec{H}$$

We consider monochromatic waves

$$\vec{E}(\vec{r}, t) = \vec{E}(\vec{r}) e^{i\omega t}$$

$$\vec{B}(\vec{r}, t) = \vec{B}(\vec{r}) e^{i\omega t}$$

$$\vec{E}(\vec{r}, t) = \frac{1}{\sqrt{2\pi}} \int e^{-i\omega t} \vec{E}(\omega) d\omega$$

$$\partial_t \vec{E}(\vec{r}, t) = \frac{1}{\sqrt{2\pi}} \int (i\omega) e^{-i\omega t} \vec{E}(\omega) d\omega$$

$\vec{E}(\vec{r}, t)$  is real

$$\Rightarrow E^*(\vec{r}, t) = E(\vec{r}, t)$$

$$= \int e^{i\omega t} \vec{E}^*(\omega) d\omega = \int e^{-i\omega t} E(\omega) d\omega$$

$$\int e^{-i\omega t} \vec{E}^*(-\omega) d\omega$$

$$\Rightarrow E^*(-\omega) = E(\omega)$$

For a monochromatic wave we have

$$E = (e^{-i\omega t} E(\omega) + e^{i\omega t} E^*(-\omega)) \frac{1}{\sqrt{2\pi}} =$$

$$= \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \text{Re} E(\omega) e^{-i\omega t}$$

Maxwell eq.

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$$\vec{\nabla} \times \vec{H} = \frac{1}{c} \partial_t \vec{D} + \frac{4\pi}{c} \vec{j}$$

$$\downarrow$$
$$-i\omega$$

$$\Rightarrow \vec{\nabla} \cdot \vec{D} = 4\pi \rho$$

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c} \partial_t \vec{B}$$

$$\downarrow$$
$$-i\omega$$

$$\Rightarrow \vec{\nabla} \cdot \vec{B} = 0$$

material eqs

$$\vec{j} = \sigma \vec{E}$$
$$\vec{D} = \epsilon \vec{E}$$
$$\vec{B} = \mu \vec{H}$$

$$\begin{aligned} \vec{\nabla} \times \frac{\vec{D}}{\mu} &= \frac{1}{c} (-i\omega \epsilon + 4\pi \sigma) \vec{E} \\ &= -\frac{i\omega}{c} \left( \epsilon + \frac{4\pi \sigma c}{\omega} \right) \vec{E} \\ &\equiv \epsilon(\omega) \end{aligned}$$

conductivity  $\sigma$  described by  
the imaginary part of  $\epsilon(\omega)$

$$\Rightarrow \vec{\nabla} \times \frac{\vec{D}}{\mu} = -\frac{i\omega}{c} \epsilon \vec{E}$$

$$\vec{\nabla} \times \vec{E} = \frac{i\omega}{c} \vec{B}$$

$$\begin{aligned} \Rightarrow \vec{\nabla} \times (\vec{\nabla} \times \vec{E}) &= \frac{i\omega}{c} \mu - \frac{i\omega}{c} \epsilon \frac{\partial}{\partial t} \vec{E} \\ &= \mu \epsilon \frac{\omega^2}{c^2} \vec{E} \end{aligned}$$

$$\vec{\nabla} \times (\vec{B} \times \vec{B}) = \mu \epsilon \left( \frac{i\omega}{c} \right) \frac{i\omega}{c} \vec{B} = \mu \epsilon \frac{\omega^2}{c^2} \vec{B}$$



if there are no free charges we thus

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have the  $\nabla \cdot \vec{E} = 0$   $\nabla \cdot \vec{B} = 0$

$$-\nabla^2 \vec{E} = \frac{\omega^2}{c^2} \mu \epsilon \vec{E}$$

$$\nabla \cdot \vec{E} = 0$$

$$-\nabla^2 \vec{B} = \frac{\omega^2}{c^2} \mu \epsilon \vec{B}$$

$$\nabla \cdot \vec{B} = 0$$

$$\vec{B} = \frac{c}{i\omega} \nabla \times \vec{E}$$

$$\vec{E} = -\frac{c}{i\omega} \frac{\nabla \times \vec{B}}{\epsilon \mu}$$

Because the equations are linear

we can consider plane wave solutions

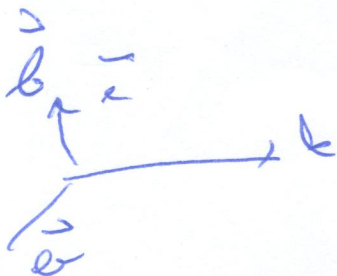
$$\vec{E} = \vec{e} e^{i\vec{k} \cdot \vec{x}} \quad \vec{B} = \vec{b} e^{i\vec{k} \cdot \vec{x}}$$

The general solution is given by a linear combination of these solutions.

$$\nabla^2 \rightarrow -k^2 \quad \Rightarrow \quad k^2 = \frac{\omega^2}{c^2} \mu \epsilon$$

$$\vec{b} = \frac{c}{i\omega} i\vec{k} \times \vec{e}$$

$$\vec{e} = -\frac{c}{\omega \epsilon \mu} \vec{k} \times \vec{b}$$



Energy density

$$\vec{E}_r = \text{Re } \vec{e}_\lambda e^{i\vec{k}\cdot\vec{x} - i\omega t} = \text{Re } \vec{E}$$

$$\vec{B}_r = \text{Re } \left( \frac{c\vec{k}}{\omega} \times \vec{e}_\lambda \right) e^{i\vec{k}\cdot\vec{x} - i\omega t} = \text{Re } \vec{B}$$

energy density  $u = \epsilon \frac{E_r^2}{8\pi} + \frac{B_r^2}{8\pi\mu}$

$$= \epsilon \frac{(\vec{E} + \vec{E}^*)^2}{4 \cdot 8\pi} + \frac{(\vec{B} + \vec{B}^*)^2}{4 \cdot 8\pi}$$

$$\overline{E_r^2} \Rightarrow \overline{E_r^2} = 0 \quad \overline{E_r^{*2}} = 0$$
$$\overline{B_r^2} = 0 \quad \overline{B_r^{*2}} = 0$$

$$= u = \frac{\epsilon}{16\pi} \vec{E} \cdot \vec{E}^* + \frac{\vec{B} \cdot \vec{B}}{16\pi\mu}$$

$$= \left( \frac{\epsilon}{16\pi} + \frac{c^2 \mu^2}{\omega^2} \frac{1}{16\pi\mu} \right) \vec{E} \cdot \vec{E}^* = \frac{\epsilon}{8\pi} \vec{e}_\lambda \cdot \vec{e}_\lambda^*$$

$$k^2 = \frac{\omega^2}{c^2} \mu \epsilon$$

Energy flow

$$\vec{p} = \frac{c}{4\pi} \vec{E}_r \times \vec{H}_r$$

$$= \frac{c}{4\pi} \left( \frac{\vec{E} + \vec{E}^*}{2} \right) \times \left( \frac{\vec{H} + \vec{H}^*}{2} \right)$$

$$e^{\pm 2i\omega t} = 0$$

$$\Rightarrow \vec{p} = \frac{c}{16\pi} (\vec{E} \times \vec{H}^* + \vec{E}^* \times \vec{H})$$

$$\vec{E} \times \vec{H}^* = \vec{E} \times \frac{\vec{B}^*}{\mu} = \vec{E} \times \left( \frac{c}{\omega} \vec{e}_\lambda \times \vec{E}^* \right)$$

$$\vec{E}^* \times \vec{H} = \vec{E}^* \times \left( \frac{\omega\mu}{c} \vec{e}_\lambda \times \vec{E} \right)$$



$$\begin{aligned} \vec{E} \times \vec{E}^* &= \vec{e} e^{i\vec{n}\cdot\vec{x}} (\vec{e}^* e^{-i\vec{n}\cdot\vec{x}}) e^{-i\omega t} \\ &= \vec{e} \times (\vec{e}^* e^{-i\omega t}) \\ &= (\vec{e} \cdot \vec{e}^*) \vec{e} - (\vec{e} \cdot \vec{e}) \vec{e}^* \end{aligned} \quad (72)$$

$$\begin{aligned} \vec{E}^* \times (\vec{e} \times \vec{E}) &= \vec{e}^* \times (\vec{e} \times \vec{e}) \\ &= (\vec{e}^* \cdot \vec{e}) \vec{e} - (\vec{e}^* \cdot \vec{e}) \vec{e} \end{aligned}$$

$$\Rightarrow P = \frac{c}{8\pi} (\vec{E}^* \times \vec{E})$$

$$= \frac{c}{8\pi} (\vec{e}^* \times (\frac{c\vec{n}}{\omega} \times e_{\lambda})) \frac{1}{\mu}$$

$$= \frac{\epsilon}{8\pi} \frac{c^2 k}{\epsilon \mu \omega} \vec{e}^* \cdot \vec{e}_{\lambda} = \bar{u} \frac{c}{\sqrt{\epsilon \mu}} \hat{k}$$

note that  $\bar{u} = \frac{\epsilon}{8\pi} \vec{e}_{\lambda} \cdot \vec{e}_{\lambda}^*$

$$z = \frac{\omega}{c} \sqrt{\mu \epsilon}$$