Electromotive Force

magnetic Flux $\quad \Phi = \int_S \vec{B} \cdot d\vec{s}$

Flux is independent of the surface

prove

$$\int_V \operatorname{div} \vec{B} \, d^3r = 0 \quad \Rightarrow \int_{\partial V} \vec{B} \cdot d\vec{s} = 0$$

$$\Rightarrow \int_{S_1} \vec{B} \cdot d\vec{s} + \int_{S_2} \vec{B} \cdot d\vec{s} = 0$$

if we choose the normal in the same direction this gives

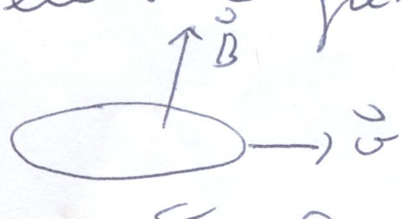
$$\int_{S_1} \vec{B} \cdot d\vec{s} - \int_{S_2} \vec{B} \, d\vec{s} = 0$$

qed

An electromotive force occurs if a conductor moves ~~through~~ a magnetic field. The reason is that the electrons feel the Lorentz force

$$F_L = e \frac{\vec{v}}{c} \times \vec{B}$$

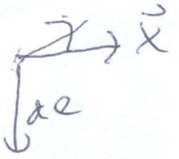
we can also interpret this as an effective electric field $\quad \vec{E}_{eff} = \frac{\vec{v}}{c} \times \vec{B}$

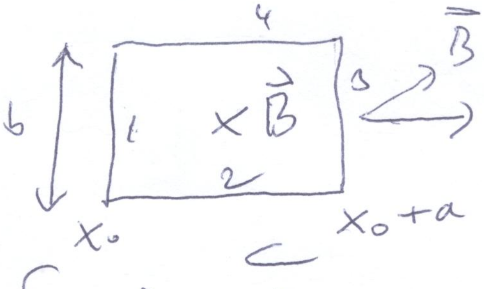


Electromotive force

$$\mathcal{E}_{ef} = \oint_C \vec{E} \cdot d\vec{l} = \int \frac{\vec{v}}{c} \times \vec{B} \cdot d\vec{l}$$

Example





$\vec{B} = \text{constant}$

$$\frac{1}{c}\int_C \vec{v}\times\vec{B}\cdot d\ell = \frac{1}{c}\frac{d}{dt}\int_C \dot{\vec{x}}\times\vec{B}\cdot\vec{d\ell}$$

$$= \frac{1}{c}\frac{d}{dt}\int_C \vec{B}\cdot d\vec{\ell}\times\dot{\vec{x}}$$

integral only nonzero on vertical sides

$$= \frac{1}{c}\frac{d}{dt}\Big(\underbrace{\int_1 \vec{B}\cdot d\vec{\ell}\times\dot{\vec{x}}}_{Bb\,x_0} + \underbrace{\int_3 \vec{B}\cdot d\vec{\ell}\times\dot{\vec{x}}}_{-\,Bb\,(x_0+a)}$$

$$= -\frac{1}{c}\frac{d}{dt}BA = -\frac{1}{c}\frac{d}{dt}\underbrace{\Phi}_{\text{flux}}$$

Next we will prove this in general

$$\mathcal{E}_{ef} = -\frac{1}{c}\frac{d\Phi}{dt}$$

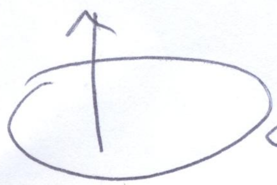
the flux changes due to a moving circuit.

<u>Faraday law</u> same formula but then the changes due to the time dependence of the magnetic field.

Flux $\phi = \int \vec{B}\cdot d\vec{s} = \int \vec{A}\cdot d\vec{l}$

$$\vec{B} = \vec{\nabla}\times\vec{A}$$

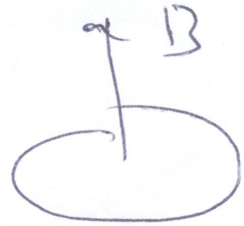
B



C_1

t

B



$C_2 = C_1 + \delta C_1$

C_1 : $\vec{x}(\tau)$ $\qquad x(0) = x(1)$

C_2 : $\vec{x}(\tau) + \delta x(\tau)$ $\qquad 0 \le \tau < 1$

$$d\vec{l} = \frac{d\vec{x}}{d\tau}\,d\tau$$

$$\delta\phi = \oint_{C_2} \vec{A}\cdot d\vec{l} - \oint_{C_1} \vec{A}\cdot d\vec{l}$$

$$= \int_0^1 d\tau\, A_i(x(\tau) + \delta x(\tau))\, \frac{d(\vec{x} + \delta x)_i}{d\tau}$$

$$- \int_0^1 d\tau\, A_i(x(\tau)) \cdot \frac{d x_i(\tau)}{d\tau}$$

$$= \int_0^1 d\tau \left(\partial_k A_i(\vec{x})\, \delta x_k \frac{dx_i}{d\tau} + A_i \frac{d(\delta x)_i}{d\tau}\right)$$

// partial integration

$$\left(-\frac{d}{d\tau} A_i\right)(\delta x_i)$$

$$= -\frac{dA_i}{dx_p}\frac{dx_p}{d\tau}\delta x_i$$

$$\Rightarrow \delta\phi = \int_0^1 d\tau \left(\partial_k A_i - \partial_i A_k\right) \times \delta x_k \frac{dx_i}{d\tau}$$

$d\tau \dfrac{d\vec{x}}{d\tau} = \vec{d\ell}$

$$\Rightarrow \delta\phi = \int_S' (\vec{d\ell})_i \, (\partial_k A_i - \partial_i A_k) \cdot \delta x_k$$

Let us look at $(\delta x \times \vec{B})_i = (\delta x \times (\vec{\nabla} \times \vec{A}))_i$

$$= \varepsilon_{ijk} \, \delta x_j \, \varepsilon_{kpq} \, \partial_{x_p} A_q$$

$$= (\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}) \, \delta x_j \, \partial_{x_p} A_q$$

$$= \delta x_q \, \partial_{x_i} A_q - \delta x_j \, \partial_{x_j} A_i$$

So $\delta\phi = -\int_S' (\delta\vec{x} \times \vec{B}) \cdot \vec{d\ell}$

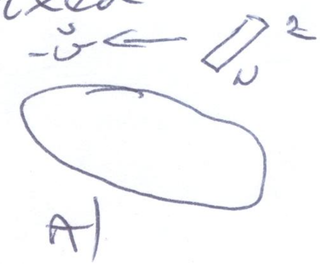
$$\Rightarrow -\frac{1}{c}\frac{\delta\phi}{\delta t} = +\frac{1}{c}\int \left(\frac{\delta\vec{x}}{\delta t} \times \vec{B}\right) \cdot \vec{d\ell}$$

$$= \frac{1}{c}\int \frac{\vec{v}}{c} \times \vec{B} \cdot d\ell$$

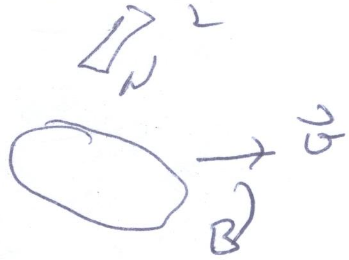
$$= \mathcal{E}_f.$$

Faraday's law

move the magnet and keep the circuit fixed



A)



B)

Faraday's law

$$\mathcal{E} = -\frac{d\phi}{dt}$$

$$\oint_C \vec{E}_{eff}\cdot d\vec{\ell} = -\frac{d}{dt}\int \vec{B}\cdot d\vec{A}$$

$\parallel$ stoke

$$\int \vec{\nabla}\times\vec{E}\cdot d\vec{A}$$

True for arbitrary loops

$$\Rightarrow \quad \vec{\nabla}\times\vec{E} = -\frac{d}{dt}\vec{B}$$

differential form of Faraday's law.

Magnetic energy

a magnetic field acting on a current transfers energy to the current.
The loss of energy of the field per second is given by

$$-\frac{dW}{dt} = \int d^3r\, \vec{E}\cdot\vec{J} = \int d^3r\, \vec{E}\cdot\frac{c}{4\pi}\vec{\nabla}\times\vec{H}$$

$$= \int d^3r\, \frac{c}{4\pi}\vec{H}\cdot\vec{\nabla}\times\vec{E}$$

$$= -\frac{c}{4\pi}\int d^3r\, \partial_t\vec{B}\cdot\vec{H}$$

$B=\mu H$

$$= -\frac{c}{8\pi}\partial_t\int d^3r\,(\vec{B}\cdot\vec{H})$$

$$\Rightarrow W = \frac{1}{8\pi}\int d^3r\, \vec{B}\cdot\vec{H}$$

$$dW = -\sum_i q_i \vec{E}_i\cdot d\vec{s}$$

$$\frac{dW}{dt} = -\sum_i q_i \vec{E}_i\,\frac{d\vec{s}}{dt}$$

$$= -\int d^3r\, \vec{E}(r)\underbrace{\rho(r)\vec{v}}_{J}$$

5 II

Galileon invariance Physics in frame A) and in frame B) is the same

$$Q = \int I \, dt = \int \frac{\mathcal{E}}{R} dt$$

same amount of charge is transferred.

$$\Rightarrow \mathcal{E}_A = \mathcal{E}_B$$

Faraday's law $\qquad \mathcal{E} = -\frac{d\phi}{dt}$

Paradox

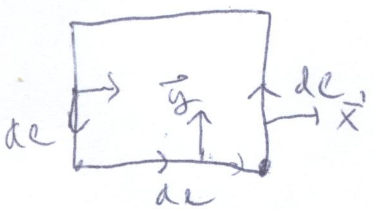
$\vec{B}$ = constant

$$\mathcal{E} = \frac{1}{c} \oint \vec{v} \times \vec{B} \cdot d\vec{l}$$

$$= \frac{1}{c} \oint \frac{d\vec{x}}{dt} \times \vec{B} \cdot d\vec{l}$$

$$= \frac{1}{c} \oint d\vec{l} \times \frac{d\vec{x}}{dt} \cdot \vec{B}$$

$$= \frac{d}{dt} \frac{1}{c} \underbrace{\oint d\vec{l} \times \vec{x}}_{2A} \cdot \vec{B}$$



wrong

solution the contour also depends on t.

parameterication of c $\qquad x(\tau),\ 0\le\tau<1$

$x(0) = x(1)$

$$\mathcal{E} = \frac{1}{c}\int_0^1 \underbrace{d\tau\,\frac{d\vec{x}}{d\tau}}_{\vec{dl}} \times \frac{d\vec{x}}{dt}\cdot\vec{B}$$

$$= \frac{1}{c}\frac{d}{dt}\int_0^1 d\tau\,\frac{d\vec{x}}{d\tau}\times\vec{x}\cdot\vec{B} - \underbrace{\frac{1}{c}\int_0^1 d\tau\,\frac{d}{d\tau}\dot{\vec{x}}\times\vec{x}\cdot\vec{B}}$$

$$\frac{1}{c}\int_0^1 d\tau\,\dot{\vec{x}}\times\frac{d\vec{x}}{d\tau}\cdot\vec{B}$$

$$= \frac{1}{c}\int_c \vec{v}\times\vec{dl}\cdot\vec{B}$$

$$= \frac{1}{c}\int \vec{v}\times\vec{B}\cdot\vec{dl}$$

$$\Rightarrow \frac{2}{c}\oint\vec{v}\times\vec{B}\cdot\vec{dl} = \frac{1}{c}\frac{d}{dt}\int_0^1 d\tau\,\frac{d\vec{x}}{d\tau}\times\vec{x}\cdot\vec{B}$$

Maxwell equation

1) $\vec{\nabla} \times \vec{E} = -\frac{1}{c}\partial_t B$

2) $\vec{\nabla} \cdot \vec{E} = 4\pi\rho$

3) $\vec{\nabla} \cdot \vec{B} = 0$

4) $\vec{\nabla} \times \vec{B} = \frac{4\pi}{c}j + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$

We will now argue that this term should be there

1) and 3) are consistent

$$\underbrace{\vec{\nabla} \cdot \vec{\nabla} \times \vec{E}}_{=0} = -\frac{1}{c}\partial_t \vec{\nabla} \vec{B} = 0$$

Continuity eq $\partial_t \rho + \vec{\nabla} \vec{j} = 0$

$$\partial_t \rho \neq 0 \Rightarrow \vec{\nabla} \vec{j} \neq 0$$

and $\underbrace{\vec{\nabla} \cdot (\vec{\nabla} \times \vec{B})}_{=0} = \frac{4\pi}{c}\vec{\nabla} \cdot \vec{j} \neq 0$

Maxwell solved this by adding an extra term to 4

$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c}(\vec{j} + \vec{j}_D)$$

displacement current

To be consistent, we need that

$$\vec{\nabla} \cdot (\vec{j} + \vec{j}_D) = 0$$

$$\Rightarrow \vec{\nabla} \vec{j}_D = -\vec{\nabla} \vec{j} = + \frac{\partial \rho}{\partial t}$$

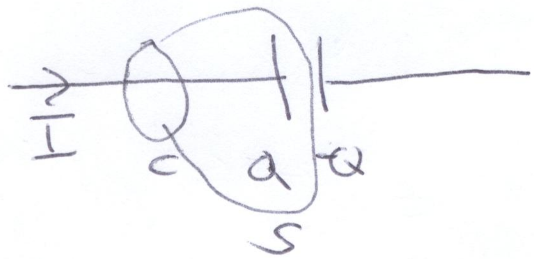
$$= \frac{1}{4\pi} \vec{\nabla} \frac{\partial \vec{E}}{\partial t}$$

$$\Rightarrow \vec{j}_D = \frac{1}{4\pi} \frac{\partial \vec{E}}{\partial t} + \vec{\nabla} \times \vec{Q}$$

The simplest choice $\vec{Q} = 0$ appeared to be the right choice.

$$\Rightarrow \text{ME 4)}: \quad \vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{j} + \frac{1}{c} \partial_t \vec{E}$$

Example of displacement current



$$\oint_C \vec{B} \cdot dl = \frac{4\pi}{c} I$$

$$\shortparallel$$

$$\int_S \vec{\nabla} \times \vec{B} \cdot \vec{dS}$$

choose S through capacitor.

~~which~~ without displacement current we have that $\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{j} = 0$

with displacement current

$$\vec{\nabla} \times \vec{B} = \underbrace{\frac{4\pi}{c} \vec{j}}_{=0} + \frac{1}{c} \partial_t \vec{E}$$

$$\Rightarrow \int_S \vec{\nabla} \times \vec{B} \cdot dS = \int_S \frac{1}{c} \partial_t \vec{E} \cdot dS$$

$$= \frac{1}{c} \partial_t \int_S \vec{E} \cdot dS$$

We can close S because $\vec{E} = 0$ inside C

$$= \frac{1}{c} \partial_t \int_V \underbrace{\mathrm{div}}_{4\pi\rho} \vec{E} \cdot dV$$

$$= \frac{1}{c} 4\pi \, \partial_t Q = \frac{4\pi}{c} I$$ okay

Magnetic energy

a magnetic field acting on a current transfers energy to the current.
The loss of energy of the field per second is given by

$$-\frac{dW}{dt} = \int d^3r\ \vec{E}\cdot\vec{J} = \int d^3r\ \vec{E}\cdot\frac{c}{4\pi}\vec{\nabla}\times\vec{H}$$

$$= \int d^3r\ \frac{c}{4\pi}\ \vec{H}\cdot\vec{\nabla}\times\vec{E}$$

$$= -\frac{c}{4\pi}\int d^3r\ \partial_t\vec{B}\cdot\vec{H}$$

$B = \mu H$

$$= -\frac{c}{8\pi}\partial_t\int d^3r(\vec{B}\cdot\vec{H})$$

$$\Rightarrow W = \frac{1}{8\pi}\int d^3r\ \vec{B}\cdot\vec{H}$$

$$dW = -\sum_i q_i\vec{E}_i\cdot d\vec{s}$$

$$\frac{dW}{dt} = -\sum_i q_i\vec{E}_i\ \frac{d\vec{s}}{dt}$$

$$= -\int d^3r\ \vec{E}(r)\underbrace{\rho(r)\vec{v}}_{J}$$

We can rewrite this in terms of the vector potential:

$$W = \frac{1}{8\pi}\int d^3r\ \vec{B}\cdot\vec{H} = \frac{1}{8\pi}\int d^3r\ \vec{\nabla}\times\vec{A}\cdot\vec{H}$$

$$= \frac{1}{8\pi}\int d^3r\ \varepsilon_{ijk}\,\partial_j A_k\cdot H_i$$

$$= -\frac{1}{8\pi}\int d^3r\ \varepsilon_{ijk}\,A_k\,\partial_j H_i$$

$$= \frac{1}{8\pi}\int d^3r\ \varepsilon_{kji}\,A_k\,\partial_j H_i$$

$$= \frac{1}{8\pi}\int d^3r\ A\cdot\underbrace{\vec{\nabla}\times\vec{H}}_{\frac{4\pi}{c}\vec{j}}$$

$$W = \frac{1}{2c}\int d^3r\ \vec{A}\cdot\vec{j}$$

For a circuit this becomes $d^3r\,\vec{j} = I\,d\ell$

$$W = \frac{1}{2c}\oint \vec{A}\cdot I\,d\vec{\ell}$$

$$= \frac{1}{2c}\int_S I\,\underbrace{\vec{\nabla}\times\vec{A}}_{B}\cdot d\vec{A} = \frac{1}{2c}\,I\,\underset{\text{flux}}{\Phi}$$

magnetic field depend linearly on the currents $\Rightarrow$ $\Phi_i = c\sum_j \underset{\uparrow}{L_{ij}}\,I_j$

coefficients of inductio

$$\Rightarrow W = \frac{1}{2}\sum I_i\,L_{ij}\,I_j$$

Momentum tensor

$$\vec F = \frac{1}{c}\int_V d^3r\, \vec j \times \vec B \qquad \vec j = \sum_i q_i \vec v_i\, \delta(\vec r - \vec r_i)$$

force
of field on current

$$\vec\nabla \times \vec B = \frac{4\pi}{c}\vec j$$

$$\Rightarrow F = \frac{1}{4\pi}\int_V d^3r\, \underbrace{(\vec\nabla \times \vec B) \times \vec B}_{-\vec B \times (\vec\nabla \times B)}$$

$$F_k = -\frac{1}{4\pi}\int_V d^3r\, B_i \partial_k B_i - B_i \partial_i B_k$$

$$= -\frac{1}{4\pi}\int_V d^3r\, \frac{1}{2}\partial_k \vec B^2 - \partial_i (B_i B_k)$$

note that $\partial_i B_i = 0$

$$= \frac{1}{4\pi}\int_V d^3r\, \partial_i \left(B_i B_k - \frac{1}{2}\delta_{ik}\vec B^2\right)$$

$$= \int_{\partial V} da\, n_i T_{ik}$$

momentum tensor $T_{ik} = \frac{1}{4\pi}\left(B_i B_k - \frac{1}{2}\delta_{ik}\vec B^2\right)$

Force on charges and current

$$\vec{F} = \int d^3r \left(\rho \vec{E} + \frac{1}{c} \vec{j} \times \vec{B} \right)$$

$$\rho = \frac{1}{4\pi} \vec{\nabla} \vec{E} \qquad \vec{j} = \frac{c}{4\pi} (\vec{\nabla} \times \vec{B}) - \frac{1}{4\pi} \partial_t \vec{E}$$

$$\Rightarrow F_i = \frac{1}{4\pi} \int d^3r \left(E_i \partial_k E_k - \frac{1}{c} (\vec{E} \times \dot{B})_i + ((\vec{\nabla} \times \vec{B}) \times \vec{B})_i \right)$$

$$E_i \partial_k E_k = \partial_k (E_i E_k) + E_k \partial_i E_k - E_k \partial_k E_i - E_k \partial_i E_k$$

$$= \partial_k \left(E_i E_k - \frac{1}{2} \delta_{ik} \vec{E}^2 \right) + E_k \left(\partial_i E_k - \partial_k E_i \right)$$

$$(\vec{E} \times (\vec{\nabla} \times \vec{E}))_i = E_k \, \varepsilon_{ikp} \cdot \varepsilon_{prs} \partial_r E_s$$

$$= E_k \partial_i E_k - E \partial_k E_i$$

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c} \partial_t B$$

$$((\vec{\nabla} \times \vec{B}) \times \vec{B})_i = -\vec{B} \times (\vec{\nabla} \times \vec{B}) = - B_k \partial_i B_k + B_k \partial_k B_i$$

$$= \partial_k \left(B_k B_i - \frac{1}{2} \delta_{ik} \vec{B}^2 \right)$$

$$\Rightarrow F_i = - \int \frac{d^3r}{4\pi c} \partial_t (\vec{E} \times \vec{B}) + \int d^3r \, \partial_k T_{ik}$$

$$T_{ik} = E_i E_k - \frac{1}{2} \delta_{ik} \vec{E}^2 + B_k B_i - \frac{1}{2} \delta_{ik} \vec{B}^2$$

$$\int_V d^3r \, \partial_k T_{ik} = \int_{\partial V} dS \, n_k T_{ki}$$

$$\underbrace{F_i}_{\frac{d\vec P_{mech}}{dt}} = \underbrace{-\int \frac{d^3r}{4\pi c}\,\partial_t(\vec E\times\vec B)_i}_{-\frac{d}{dt}\vec P_{field}} + \underbrace{\int_{\partial V} ds\, n_k T_{ki}}_{\text{momentum flow throug surface}}$$

$$\Rightarrow \frac{d}{dt}\left(\vec P_{mechanical} + \vec P_{field}\right) = \int_{\partial V} ds\, n_k T_{ki}$$

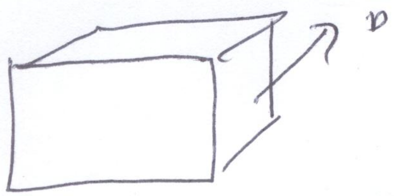
$\Rightarrow$ momentum of the em field is

$$\vec P = \frac{1}{4\pi c}\int d^2r\,(\vec E\times\vec B)$$

The angular momentum of the field is $\vec L = \frac{1}{4\pi c}\int d^3r\ \vec r\times(\vec E\times\vec B)$

Momentum density

$$\vec{p} = \frac{\vec{E}\times\vec{B}}{4\pi c}$$



c

momentum flow per unit area per second is $c\,.\,\vec{p}\,.\,\vec{n} = \frac{\vec{E}\times\vec{B}}{4\pi}\,.\,\vec{n}$

energy $E = pc$

$\Rightarrow$ energy flow per second per unit area is $\frac{\vec{E}\times\vec{B}}{4\pi}\,.\,\vec{n}$

$\vec{P} \equiv \frac{\vec{E}\times\vec{B}}{4\pi}$ is known as the Poynting vector.