Special Relativity

a) General definitions

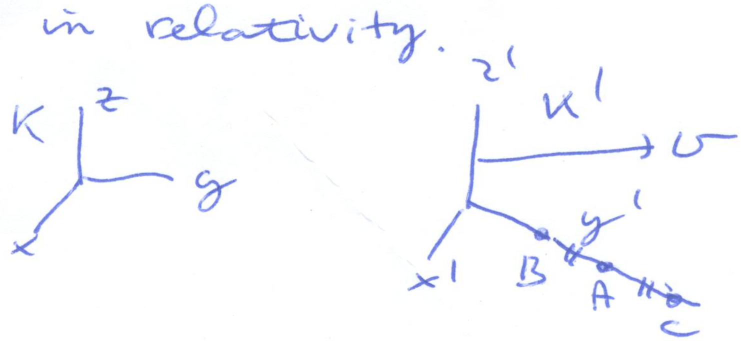
<u>inertial frame</u> a frame in which a particle on which no forces are ~~exerted~~ moves with constant velocity

<u>system of reference</u> coordinate system + clock

<u>principle of relativity</u> all laws of physics are identical in all inertial frames

<u>maximum velocity</u> The maximum velocity in all inertial frame is the speed of light.. This has been tested with great accuracy by experiments This is also known as Einsteins principle of relativity

time is absolute in Newtonian dynamics but not in relativity.

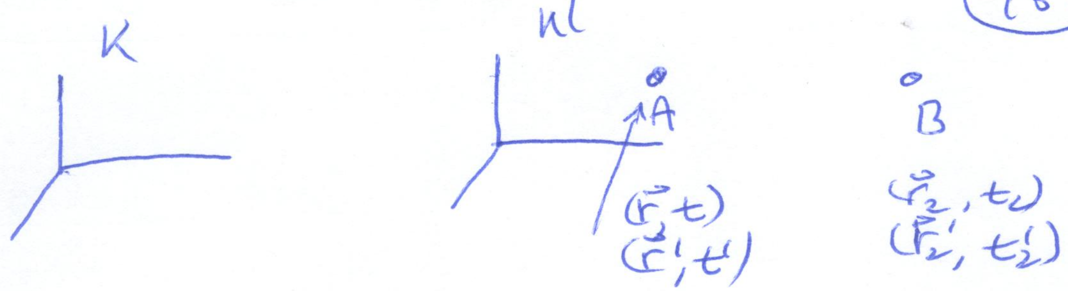


a signal emitted at A arrives simultaneously ~~in~~ at B and C in K' but not in K

<u>event</u> position + time

<u>world point</u> position + time coordinates

<u>world line</u> space time coordinates of a particle as a function of time



if A and B are connected by a light ray then

$$(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2-c^2(t_2-t_1)^2=0$$

$$(x_1'-x_2')+(y_1'-y_2')+(z_1'-z_2')-c^2(t_2'-t_1')^2=0$$

relativistic interval between two events

$$s_{12}=\left(c^2(t_2-t_1)^2-(\vec r_1-\vec r_2)^2\right)^{1/2}$$

infinitesimally: $ds^2=c^2dt^2-dx^2-dy^2-dz^2$

Theorem ds^2 is the same for all inertial frames

ok if $ds=0$

if $ds\neq 0$ then $(ds)^2=a(\vec v_1,\vec v_2)(ds')^2$

$a(v_1,v_2)$ does not depend on spacetime because of the homogeneity of space time. That is why a only depends on v_1 and v_2

consider 3 coordinate systems

$K \quad v=0 \qquad (ds_1)^2=a(v_1)\,ds^2$

$K_1 \quad v_1 \qquad (ds_2)^2=a(v_2)\,ds^2$

$K_2 \quad v_2 \qquad (ds_2)^2=a(v_1,v_2)(ds_1)^2$

$$\Rightarrow a(v_1,v_2)=\frac{a(v_2)}{a(v_1)}$$

homogeneity of space time

$\Rightarrow a(v_1, v_2)$ only depends on $(\vec{v}_1 - \vec{v}_2)^2$

$\Rightarrow$ Let $a(v_1, v_2) = b((v_1 - v_2)^2)$

$$\Rightarrow b(v_1^2 + v_2^2 - 2v_1 v_2 \cos\alpha) = \frac{b(v_2^2)}{b(v_1^2)}$$

does not depend on α

only possible if $b(v^2) =$ constant

$\Rightarrow a(v_1 v_2) =$ constant $= \alpha$

$$a(v_1, v_2) = \frac{a(v_1)}{a(v_2)} \Rightarrow \alpha = \frac{\alpha}{\alpha} = 1$$

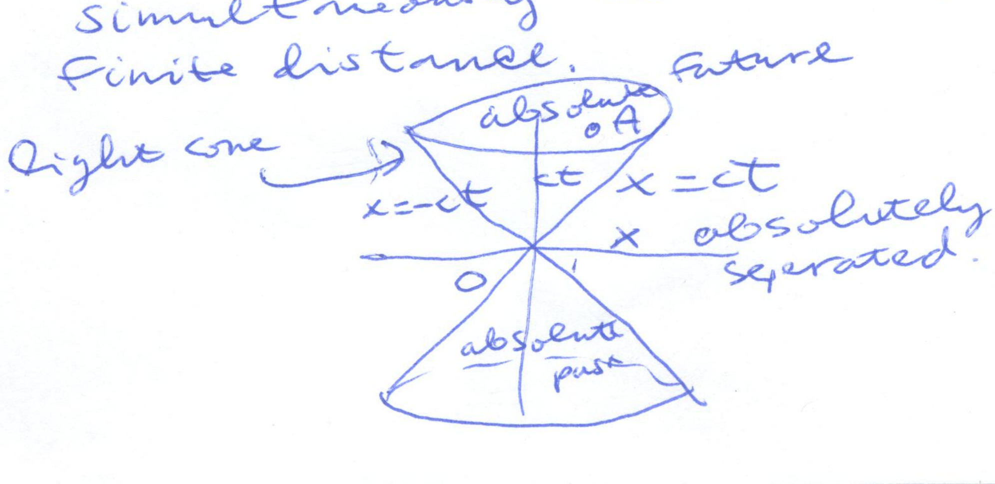
$\Rightarrow (ds)^2$ is the same for all frames

time like intervals intervals for which there exists a frame in which two events occur at the same space point. Then

$$s_{12}^2 = c^2(t_2 - t_1)^2 - (\vec{x}_2 - \vec{x}_1)^2 > 0$$

space like interval $s_{12}^2 = c^2(t_1 - t_2)^2 - (\vec{x}_1 - \vec{x}_2)^2 < 0$

$\exists$ frame in which two events occure simultaneously but are separated by a finite distance.



There is no frame for which A and O occur simultaneously $\Rightarrow$ A is in the future for all frames

<u>proper time</u> time by a clock moving with the object

in cdt the clock moves by (dx, dy, dz)

in K', frame of the clock, $dx' = dy' = dz' = 0$

$(ds)^2$ is invariant:

$$\Rightarrow c^2dt^2 - dx^2 - dy^2 - dz^2 = c^2dt'^2$$

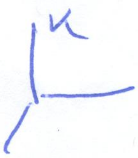
$$\Rightarrow (dt') = dt \left(1 - \frac{dx^2 + dy^2 + dz^2}{c^2dt^2}\right)^{1/2}$$

$$= dt\sqrt{1 - \frac{v^2}{c^2}}$$

for finite time intervals

$$t_2' - t_1' = \int_{t_1}^{t_2} dt\sqrt{1 - \frac{v^2}{c^2}}$$

proper time is less: moving clocks run slower



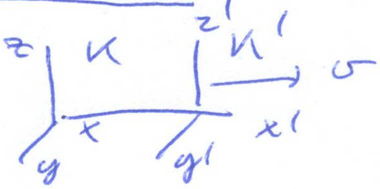
- in K clocks in K' lag
- in K' clocks in K lag.

no contradiction because we have to compare clocks in the same inertial frame

<u>Twin paradox</u>

moving twin stays younger
- no contradiction because moving twin is not an inertial frame.

Lorentz transformations



Question: event (x, y, z, t) in K, what are its coordinates in K'?

in CM $\quad t' = t, \; y' = y, \; z' = z, \; x' = x - vt$

relativity s^2 is the same for all frames

$$x_0 = ct \qquad x_0' = ct'$$
$$x_1 = x \qquad x_1' = x'$$
then $x_0^2 - x_1^2 = x_0'^2 - x_1'^2$

Homogeneity of space $\Rightarrow$ relation should be linear

$$\begin{pmatrix} x_0' \\ x_1' \end{pmatrix} = \underset{\begin{pmatrix} a & b \\ d & f \end{pmatrix}}{\underset{\shortparallel}{L(\beta)}} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \qquad \beta = \frac{v}{c}$$

$$\begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = \begin{pmatrix} x_0' \\ x_1' \end{pmatrix} \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \begin{pmatrix} x_0' \\ x_1' \end{pmatrix}$$
$$= \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} L^T(\beta) \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} L(\beta) \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}$$

$$\forall x_0, x_1 \Rightarrow \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = L^T(\beta) \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} L(\beta)$$
$$\begin{pmatrix} a & d \\ b & f \end{pmatrix} \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \begin{pmatrix} a & b \\ d & f \end{pmatrix}$$
$$= \begin{pmatrix} a^2 - d^2 & ab - df \\ ba - fd & b^2 - f^2 \end{pmatrix}$$

$$\Rightarrow \begin{array}{l} ab = fd \\ a^2 - d^2 = 1 \end{array} \qquad b^2 - f^2 = -1$$

for $x_1 = 0$ we have that $x_1' = -\beta x_0'$

$$\begin{pmatrix} x_0' \\ x_0' \cdot \beta \end{pmatrix} = \begin{pmatrix} a & b \\ d & f \end{pmatrix} \begin{pmatrix} x_0 \\ 0 \end{pmatrix} = \begin{pmatrix} a x_0 \\ d x_0 \end{pmatrix} \Rightarrow \frac{d}{a} = -\beta$$

at $x_1' = 0$ we have that $x_1 = vt = \beta x_0$

$x_1' = d x_0 + f x_1 \Rightarrow 0 = d x_0 + f\beta x_0$

$\Rightarrow d + f\beta = 0$

$\beta = -\frac{d}{a}$

$\Rightarrow$ $d + -f\frac{d}{a} = 0$

$\Rightarrow a = f$

$a^2 - d^2 = 1 \Rightarrow a^2 - \beta^2 a^2 = 1$

$\Rightarrow a = \frac{1}{\sqrt{1-\beta^2}}$

$\Rightarrow f = \frac{1}{\sqrt{1-\beta^2}}$, $d = \frac{-\beta}{\sqrt{1-\beta^2}}$

$b^2 = f^2 - 1 = \frac{1}{1-\beta^2} - \frac{(1-\beta^2)}{1-\beta^2} = \frac{\beta^2}{(1-\beta^2)}$

$\Rightarrow b = -\frac{\beta}{\sqrt{1-\beta^2}}$ sign follows from nonrelativistic limit

$$\Rightarrow L = \begin{pmatrix} \frac{1}{\sqrt{1-\beta^2}} & \frac{-\beta}{\sqrt{1-\beta^2}} \\ -\frac{\beta}{\sqrt{1-\beta^2}} & \frac{1}{\sqrt{1-\beta^2}} \end{pmatrix}$$

Useful parameterization $\beta = \tanh y$

$\uparrow$ rapidity

then $\gamma = \dfrac{1}{\sqrt{1-\beta^2}} = \cosh y$

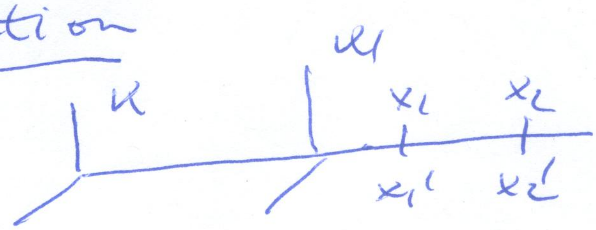
$$\beta\gamma = \sinh y$$

$$L = \begin{pmatrix} \cosh y & -\sinh y \\ \sinh y & \cosh y \end{pmatrix}$$

Lorentz transformation

$$x_0' = \gamma x_0 - \beta\gamma x_1$$
$$x_1' = -\beta\gamma x_0 + \gamma x_1$$
$$x_2' = x_2$$
$$x_3' = x_3$$

Lorentz contraction



length of rod in K: $l_0 = x_2 - x_1$

measure the length of the rod at $ct_1' = ct_2' = 0$ (the times have to be the same for the measurement of the rod)

$$v_y = \frac{v_y'}{\gamma\left(1 + \frac{v\, v_x'}{c^2}\right)}$$

same for $v_z = \dfrac{v_z'}{\gamma\left(1 + \frac{v V_x'}{c^2}\right)}$

classical limit $\quad v_x = v_x' + v$

$v_y = v_y'$

$v_z = v_z'$

$ct_1' = \gamma(ct_1 - \beta x_1) \Rightarrow ct_1 = \beta x_1$

$ct_2' = \gamma(ct_2 - \beta x_2) \Rightarrow ct_2 = \beta x_2$

$x_1 = \gamma(-\beta c t_1 + x_1) = \gamma(-\beta^2 x_1 + x_1)$

$x_2' = \gamma(-\beta c t_2 + x_2) = \gamma(-\beta^2 x_2 + x_2)$

$\Rightarrow x_1' = \frac{x_1}{\gamma} \quad \Rightarrow (x_2' - x_1') = \frac{(x_2 - x_1)}{\gamma}$

$x_2' = \frac{x_2}{\gamma} \qquad l = \frac{l_0}{\gamma}$ ← proper length

Lorentz contraction

for volume: $V = V_0 \sqrt{1-\beta^2}$

no contraction $\perp$ $\vec{v}$

Addition of velocities

$v_x = \frac{dx}{dt} \quad v_x' = \frac{dx'}{dt'}$

$dx = \gamma(dx' + v\,dt') \quad dy' = dy \quad dz' = dz$

$dt = \gamma(dt' + \frac{v}{c^2} dx')$

$\Rightarrow \frac{dt}{dt'} = \gamma(1 + \frac{v}{c^2} v_x'$

$v_x = \frac{dx}{dt} = \frac{dx' + v\,dt'}{dt' + \frac{v}{c^2} dx'}$

$= \frac{v_x' + V}{1 + \frac{v V_x'}{c^2}} \quad v_y = \frac{dy}{dt} = \frac{dy'}{\gamma(dt' + \frac{v}{c^2} dx')}$

4 vectors

$(ct, x, y, z) = (x^0, x^1, x^2, x^3)$

invariant length $x_0^2 - \vec{x}^2$

Four vector a set of quantities that transforms as (x^0, x^k)

$$A^0 = \gamma\left(A^{0\prime} + \frac{v}{c} A^{1\prime}\right)$$
$$A^1 = \gamma\left(A^{1\prime} + \frac{v}{c} A^{0\prime}\right)$$
$$A^2 = A^{2\prime}$$
$$A^3 = A^{3\prime}$$

$v \parallel$ 1-axis

$A^{0\,2} - \vec{A}^2 =$ invariant

metric tensor $g^{\mu\nu} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$

$$g_{\mu\nu} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$$

$$g^{\mu\nu} g_{\nu\mu'} = \delta^{\mu}{}_{\mu'}$$

Greek indices $0, 1, 2, 3$

Latin indices $1, 2, 3$

repeated indices imply summation

contravariant vector $A^{\mu} = g^{\mu\mu'} A_{\mu'}$

covariant vector $A_{\mu} = g_{\mu\mu'} A^{\mu'}$

Scalar product

$$A_\mu B^\mu = A_0 B^0 + A_1 B^1 + A_2 B^2 + A_3 B^3$$
$$= A_0 B_0 - A_1 B_1 - A_2 B_2 - A_3 B_3$$

time like $\quad A_\mu B^\mu > 0$

space like $\quad A_\mu B^\mu < 0$

null vector $\quad A_\mu B^\mu = 0$

notation $\quad A^\mu = (A_0, \vec{A})$

$\quad x^\mu = (ct, \vec{r})$

<u>tensor</u> is defined by its transformation properties

$$A'^{\mu'\nu'} = L^{\mu'}{}_{\mu} L^{\nu'}{}_{\nu} A^{\mu\nu}$$

Lorentz transformation

Contravariant $\quad A^{\mu\nu}$

covariant $\quad A_{\mu\nu}$

mixed $\quad A_\mu{}^\nu \quad A^\mu{}_\nu$

$$A^\mu{}_\nu = g_{\nu\nu'} A^{\mu\nu'}$$

$$A_{\mu\nu} = g_{\mu\mu'} g_{\nu\nu'} A^{\mu'\nu'}$$

Symmetric tensor $\quad A_{\mu\nu} = A_{\nu\mu}$

Anti-symmetric tensor $\quad A_{\mu\nu} = -A_{\nu\mu}$

Trace of tensor $A^{\mu}{}_{\mu} = A^0{}_0 + A^1{}_1 + A^2{}_2 + A^3{}_3$

$$\delta_\mu{}^\nu = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$$

$$g_{\mu\nu} = g_{\mu\nu'} \delta^{\nu'}{}_{\nu} \qquad g_{\mu\nu} = g^{\mu\nu}$$

Group theoretical analysis of Lorentz transformation

invariant $A_\mu A^\mu = A'_\mu A^{\mu'}$

$$\Rightarrow A'_\mu A^{\mu\prime} = A'^{\,\mu'} g_{\mu'\mu} A'^{\mu}$$

$$= L^{\mu'}{}_{\mu''} A^{\mu''} g_{\mu'\mu} L^{\mu}{}_{\mu'''} A^{\mu'''}$$

$$= A^{\mu'} g_{\mu'\mu} A^{\mu}$$

True for all A_μ

$$\Rightarrow g_{\mu''\mu'''} = \underbrace{L^{\mu'}{}_{\mu''}}_{= (L^T)_{\mu''}{}^{\mu'}} g_{\mu'\mu} L^{\mu}{}_{\mu'''}$$

$$\Rightarrow \boxed{g = L^T g L}$$

$\det L^T = \det L$

$\Rightarrow (\det L)^2 = 1$

$\Rightarrow \det L = \pm$

proper Lorentz transformations are those that can continuously be transformed to the identity $\Rightarrow \det L = 1$

This cannot be done for improper Lorentz transformations

$\det A = -1 \Leftarrow$ improper

$\det A = 1 \Leftarrow$ proper.

reverse is not true, eg $A = -1$ then $\det A = 1$ but A is improper.

<u>Lorentz group</u> $L^T g L = g$

set of matrices that satisfy this relation is the Lorentz group.

$g^T = g \Rightarrow \exists$ 10 independent relations

L has 16 parameters $\Rightarrow$ only 10 are independent.

Lecture #36

Lorentz group $L^T g L = g$

Finite group elements can be construct from those infinitesimally close to the identity

$$L = 1 + \varepsilon G$$

$$\left(1 + \frac{\varphi}{N} G\right)^N \xrightarrow[N \to \infty]{} e^{\varphi G}$$

$$A_\mu = g_{\mu\nu} A^\nu \qquad g_{\mu\nu} = g^{\mu\nu} \qquad g_\mu{}^\nu = \delta_\mu{}^\nu$$

$$g = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

Today
- construction of Lorentz group
- invariant tensors

Proof that it's a group

- identity satisfies this relation
- inverse satisfies the relation

$$L^T g L = g \Rightarrow L^{-1} = g L^T g$$

$$\Rightarrow (L^{-1})^T g L^{-1} = g L g \, g L^{-1} = g$$

- group property: if L_1 and L_2 are Lorentz then also $L_1 L_2$

$$L_1^T g L_1 = g \qquad L_2^T g L_2 = g$$

then $(L_1 L_2)^T g L_1 L_2 = L_2^T L_1^T g L_1 L_2$

$$= L_2^T g L_2 = g$$

The Lorentz group is continuous group. Therefore a large Lorentz transformation can be obtained by multiplying infinitesimal Lorentz transformations

$$L = 1 + \varepsilon A \Rightarrow L_{big} = \left(1 + \frac{y}{N} A\right)^N = e^{yA}$$

$\Rightarrow$ we only have to study the infinitesimal Lorentz transformations.

$L^{T} g L = g \Rightarrow A^{T} g + g A = 0$ to $O(\varepsilon)$

$\Rightarrow A^{T} = -g A g$

$\Rightarrow (A^{T})_{\alpha\rho} = -g_{\alpha\alpha} A_{\alpha\beta} g_{\beta\rho}$

$\quad \overset{\parallel}{A_{\beta\alpha}} \qquad \Rightarrow A_{\alpha\alpha} = 0$

$$A = \begin{pmatrix} 0 & A_{01} & A_{02} & A_{03} \\ A_{01} & 0 & A_{12} & A_{13} \\ A_{02} & -A_{12} & 0 & A_{23} \\ A_{03} & -A_{13} & -A_{23} & 0 \end{pmatrix}$$

$\Rightarrow$ only 6 independent parameters

<u>generators</u> set of basis matrices

$$S_1 = \begin{pmatrix} 0 & 0 \\ 0 & \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} \end{pmatrix} \quad S_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad S_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$K_1 = \begin{pmatrix} \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} & 0 \\ 0 & 0 \end{pmatrix} \quad K_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad K_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & & & \\ 0 & & 0 & \\ 1 & & & \end{pmatrix}$$

commutation relations

$[S_i, S_j] = \varepsilon_{ijk} S_k$ (rotation group)

$[K_i, K_j] = -\varepsilon_{ijk} S_k$

$[S_i, K_j] = \varepsilon_{ijk} K_k$

we write $A = -\vec{\omega} \cdot \vec{S} - \vec{\zeta} \cdot K$

Large Lorentz transformations

$$L = e^{-\vec{\omega}\cdot\vec{S} - \varphi\cdot\vec{K}}$$

rotatantion about x axis

$$e^{-\omega_1 S_1} = 1 - \omega_1 \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & -1 \\ & & 1 & 0 \end{pmatrix} + \frac{1}{2}\omega_1^2 \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & -1 & 0 \\ & & 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1-\frac{\omega_1^2}{2} & -\omega_1 \\ & & \omega_1 & 1-\frac{1}{2}\omega_1^2 \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \cos\omega_1 & -\sin\omega_1 \\ & & \sin\omega_1 & \cos\omega_1 \end{pmatrix}$$

Special Lorentz transformation

$$e^{-\varphi_1 K_1} = 1 + \varphi_1 \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} + \frac{1}{2}\varphi_1^2 \begin{pmatrix} 1 & 0 & & \\ 0 & 1 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1+\frac{1}{2}\varphi_1^2 & -\varphi_1 & & \\ -\varphi_1 & 1+\frac{1}{2}\varphi_1^2 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \to \begin{pmatrix} \cosh\varphi_1 & -\sinh\varphi_1 & & \\ -\sinh\varphi_1 & \cosh\varphi_1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

Invariant tensors

$$g'_{\mu'\nu'} = L_{\mu'}{}^{\mu} L_{\nu'}{}^{\nu} g_{\mu\nu} = (L g L^T)_{\mu'\nu'}$$

$$L^T g L = g \Rightarrow L^T = g L^{-1} g^{-1} = g L^{-1} g$$

$$\Rightarrow L g L^T = L g^2 L^{-1} g = g$$

$\Rightarrow$ g is invariant

$\Rightarrow$ $\delta^\mu{}_\nu = g^{\mu\nu'} g_{\nu'\mu}$ is invariant

Lecture # 39

$L^T g L \quad g \qquad L = 1 + \varepsilon G$

$$G = -\vec{\omega}\,\vec{S} - \vec{y}\cdot\vec{K}$$

$$L = e^{-\vec{\omega}\,\vec{S} - \vec{y}\cdot\vec{K}}$$

symmetry is preserved under Lorentz transfo

- Today invariant tensors
- Kinematics

A tensor that is symmetric in one frame is symmetric in all frames.

$$A'_{\mu\nu} = L_\mu{}^{\mu'} L_\nu{}^{\nu'} A_{\mu'\nu'}$$

$$\qquad\qquad\qquad\quad \| $$

$$\qquad\qquad\qquad A_{\nu'\mu'}$$

$$= L_\nu{}^{\nu'} L_\mu{}^{\mu'} A_{\nu'\mu'} = A'_{\nu\mu}$$

same is $A_{\mu\nu} = -A_{\nu\mu}$ then $A'_{\mu\nu} = -A'_{\nu\mu}$

$A_{\mu\nu}$ is tensor of rank 2

$A_{\mu\nu\rho}$ is tensor of rank 3

then symmetry or anti-symmetry in any of the indices is preserved

completely anti-symmetric tensor

$A_{\mu\nu\rho\eta}$ is completely antisymmetric under the interchange of any pair of indices.

This property is preserved only Lorentz transformations.

$A_{\mu\nu\rho\eta} = 0$ if two or more indices are equal

$\Rightarrow A_{\mu\nu\rho\eta}$ has only one independent component.

If $A_{0123} = 1$ then this tensor is denoted by $\varepsilon_{\mu\nu\rho\eta}$

$\varepsilon_{\mu\nu\rho\eta} = 1$ if $\mu\nu\rho\eta$ is an even permutation of 0123

$= -1$ if $\mu\nu\rho\eta$ is an odd permutation of 0123

$$\varepsilon'_{\mu'\nu'\rho'\eta'} = L_{\mu'}{}^{\mu} L_{\nu'}{}^{\nu} L_{\rho'}{}^{\rho} L_{\eta'}{}^{\eta} \varepsilon_{\mu\nu\rho\eta}$$

$\Rightarrow \varepsilon'_{0123} = \det L$

$\Rightarrow \varepsilon$ is pseudotensor.

<u>dual tensor</u> $A^{*\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma} A_{\rho\sigma}$

$\uparrow$
Hodge star

$A^{*\mu\nu} A_{\mu\nu}$ is a pseudoscalar eg if $A_{\mu\nu} = x_\mu x_\nu$ then is always contains the time component

<u>relations</u>

$$\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\alpha\beta\rho\sigma} = -2\left(\delta^{\mu}{}_{\alpha}\delta^{\nu}_{\beta} - \delta^{\mu}{}_{\beta}\delta^{\nu}_{\alpha}\right)$$

$$\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\alpha\nu\rho\sigma} = -6\,\delta^{\mu}{}_{\alpha}$$

$$\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\mu\nu\rho\sigma} = 24$$

integration

4 volume element

$$d\Omega = dx^0 dx^1 dx^2 dx^3$$

$d\Omega$ is Lorentz invariant

Gauss theorem $\int A^\mu \frac{1}{6} \varepsilon^{\mu\nu\rho\sigma} dS_{\nu\rho\sigma}$ ← 3 surface element

$$= \int \partial_\mu A^\mu d\Omega$$

Stokes theorem

$$\frac{1}{2} \int A^{\mu\nu} \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} df^{\rho\sigma} = \int dS_\mu \frac{\partial A^{\mu\nu}}{\partial x^\nu}$$

2 surface element

$$\| \quad \frac{1}{6} \varepsilon_{\mu\nu\rho\sigma} dS^{\nu\rho\sigma}$$

$$\oint A_\mu dx^\mu = \frac{1}{2} \int df^{\mu\nu} (\partial_\nu A_\mu - \partial_\mu A_\nu)$$

Four velocity

$$u^\mu = \frac{dx^\mu}{ds}$$

$$ds = c\,dt\sqrt{1-\beta^2}$$

$$u^k = \frac{dx^k}{dt}\frac{dt}{ds} = \gamma\frac{v^k}{c}$$

$$u^0 = \frac{dx^0}{ds} = \gamma$$

$$\Rightarrow u = \begin{pmatrix}\gamma \\ \gamma\frac{\vec{v}}{c}\end{pmatrix}$$

Relativistic action of a free particle

Euler Lagrangian equations

$$S = \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t)\, dt$$

generalized coordinates.

generalized momenta $p_i = \frac{\delta L}{\delta \dot{q}_i}$

eqs of motion $\delta S = 0$

$$\int_{t_1}^{t_2}\left(\frac{\delta L}{\delta q_i}\delta q_i + \frac{\delta L}{\delta \dot{q}_i}\delta\dot{q}_i\right)dt$$

partial integrati

$$\Rightarrow \frac{\delta L}{\delta q_i} - \frac{d}{dt}\frac{\delta L}{\delta \dot{q}_i} = 0$$

In relativity, the EL equations have to be scalar. Otherwise the eqs of motion would be different in different frames.

Only invariant for a free particle

ds

$$\Rightarrow \quad S = \alpha \int_a^b ds$$

We determine α from the nonrelativistic limit

$$S = \alpha \int_a^b \frac{ds}{dt}\, dt = \alpha \int_{t_1}^{t_2} \sqrt{c^2 - v^2}\, dt$$

$\Rightarrow$ Lagrange density: $L = \alpha \sqrt{c^2 - v^2}$

$$= \alpha c \left(1 - \frac{1}{2}\frac{v^2}{c^2}\right)$$

NR; $L = T - V \Rightarrow$

$$= \frac{1}{2} m v^2 - V = \alpha c - \alpha c \frac{1}{2}\frac{v^2}{c^2}$$

$$\Rightarrow \alpha = -mc \quad \Rightarrow V = mc^2$$

$$\Rightarrow \quad L = -\frac{mc^2}{\gamma}$$

Momentum and force

generalized coordinates x^μ

generalized velocities $u^\mu = \frac{dx^\mu}{ds}$

generalized momenta $p_\mu = -\frac{\delta S}{\delta x^\mu}$

$$= -\frac{\delta\left(-mc\int_a^b \sqrt{dx_\mu dx^\mu}\right)}{\delta x^\mu}$$

$$\delta S = -cm\int_a^b \frac{2}{2}\frac{dx_\mu\,\delta dx^\mu}{ds}$$

$$= -mc\int_a^b u^\mu\,\delta dx^\mu$$

$$= -mc\,u_\mu\,\delta x^\mu\Big|_a^b + mc\int_a^b \frac{du_\mu}{ds}\,\delta x^\mu\,ds$$

endpoints fixed: $\delta x^\mu\big|_a = \delta x^\mu\big|_b = 0$

$\delta S = 0$ for arbitrary variation

$$\Rightarrow \frac{du_\mu}{ds} = 0$$

To calculate the momentum keep x^μ_a fixed but keep x^μ_b variable.

$$p_\mu = -\frac{\delta S}{\delta x^\mu_b}$$

$$S = -mc\int_a^b \sqrt{dx^\mu dx_\mu} = -mc\int_a^b \frac{ds}{dx^\mu}\,dx^\mu$$

$$= -mc\int_a^b \frac{dx_\mu}{ds}\,dx^\mu = -mc\int_a^b u_\mu\,dx^\mu$$

$$\Rightarrow p_\mu = -\frac{\delta S}{\delta x^\mu_b} = +mc\,u_\mu$$

$$\vec{p} = mc\,\frac{\vec{v}}{c}\,\gamma = \gamma\,\vec{v}\,m$$

$$p_0 = mc\gamma = \frac{E}{c}$$

$\left(\frac{E}{c}, \vec{p}\right)$ is a 4 vector.

$\left(\frac{E}{c}\right)^2 - \vec{p}^2$ is invariant

for $\vec{v} = 0$ it is m^2c^2

$$\Rightarrow \left(\frac{E}{c}\right)^2 - \vec{p}^2 = m^2c^2$$

Note that $S = -mc\int\sqrt{dx^\mu dx_\mu}$

$$= -mc\int\sqrt{\frac{dx^\mu}{ds}\frac{dx_\mu}{ds}}\,ds$$

$$= -mc\int\sqrt{u^\mu u_\mu}\,ds$$

$$u_\mu u^\mu = 1$$

but variations are over trajectories for which $u_\mu u^\mu \neq 1$

- four momentum is conserved in absence of external forces
- four force $g^\mu = \frac{dp^\mu}{ds} = mc\frac{du^\mu}{ds}$

Relatistic Kinematics

$$p_\mu^{\text{initial}} = p_\mu^{\text{final}}$$

$p_\mu p^\mu = m^2 c^2$ for each particle

Four current

continuite eq. $\partial_t \rho + \vec{\nabla} \vec{J} = 0$

$$\Rightarrow \partial_{ct} (c\rho) + \partial_{x_\mu} J^\mu = 0$$

$$\Rightarrow \partial_{x^\mu} J^\mu = 0$$

$J^\mu = (c\rho, \vec{J})$

must be invariant be can-
charge is conserved

$\Rightarrow J^\mu$ is a 4-current

Covariant gauge

$\vec{\nabla} \vec{A} + \frac{1}{c} \partial_t \phi = 0$

$$\Rightarrow (\vec{\nabla}^2 - \partial_t^2)\,\phi = -4\pi\rho$$

$$(\vec{\nabla}^2 - \partial_t^2\ \vec{A} = -\frac{4\pi}{c}\vec{J}$$

$\Rightarrow (\phi, \vec{A})$ is a 4 vector

Lecture # 41

Classical action $\displaystyle\int_{x_a}^{x_b} p\,dq = \int p\dot{q}\,dt = \int m\dot{q}^2\,dt$

$$\int L\,dt \qquad L = T - V$$
$$H = T + V$$

Hamiltonian system

$$\Rightarrow \frac{\delta S}{\delta x_\mu} = P_\mu \qquad P_\mu = -\frac{\delta S}{\delta x^\mu}$$

$$L = -\frac{mc^2}{\gamma} \qquad P_\mu = mcu_\mu$$

$(\frac{E}{c}, \vec{p})$ four vector

$$\left(\frac{E}{c}\right)^2 - \vec{p}^{\,2} = m^2c^2$$

$$\frac{\partial L}{\partial \dot{x}_i} = \frac{\partial}{\partial \dot{x}_i}\left(-mc^2\sqrt{1-\frac{\dot{x}^2}{c^2}}\right) = mc^2\,\frac{1}{2}\,\frac{2\dot{x}}{c^2\sqrt{1-\frac{\dot{x}^2}{c^2}}}$$

$$= m\gamma\dot{x}$$

$$p_0 = -\frac{\partial L}{\partial c} = -\frac{\partial}{\partial c}\,{-mc^2\gamma} = 2mc\gamma$$

$(c\rho, \vec{J})$ 4-vector $\quad (\phi, \vec{A})$ 4-vector

$$L = -\frac{mc^2}{\gamma} - \frac{e}{c}\frac{u_\alpha A^\alpha}{\gamma} \qquad u_\alpha = (\gamma c, \gamma\vec{u})$$

Today: Lorentz force
EM Field tensor

Lagrangian for A_μ field

we have 4 vectors u_μ, A^μ

nontrivial scalar $u_\mu A^\mu$

Invariant Lagrangian

$$L = -\frac{mc^2}{\gamma} + \frac{a}{\gamma} u_\alpha A^\alpha$$

$$S = \int L\,dt$$

$$ds^2 = dx_0^2 - d\vec{x}^2$$

$$\Rightarrow \frac{ds}{dt} = \sqrt{c^2 - v^2}$$

$$\Rightarrow ds = c\gamma^{-1} dt$$

$\gamma^{-1} dt$ is invariant

NR limit

$$\frac{a u_0 A_0}{\gamma} = \frac{a c \gamma \phi}{\gamma} = -V$$

potential energy

$V = e\phi$

$$\Rightarrow a = -\frac{e}{c}$$

$$\Rightarrow L = -\frac{mc^2}{\gamma} - \frac{e}{c} \frac{u_\alpha A^\alpha}{\gamma} \qquad u_\alpha = (\gamma c, \gamma \vec{u})$$

Euler Lagrange equations

$$-\frac{d}{dt}\frac{\delta L}{\delta u_i} + \frac{\delta L}{\delta q_i} = 0 \qquad u_i = \dot{q}_i$$

$$\Rightarrow -\frac{d}{dt}\left(+m\gamma u^i - \frac{e}{c}A^i\right) - e\frac{\delta \phi}{\delta q_i} - \frac{e}{c}u_k\frac{\delta A^k}{\partial q_i} = 0$$

$$\Rightarrow \frac{dp_i}{dt} = \underbrace{+\frac{e}{c}\frac{\partial A^i}{\partial q_k}\underset{u_k=-u^k}{\dot{q}_k} - \frac{e}{c}\underset{-u^k}{u_k}\frac{\partial A^k}{\partial q_i}}_{\frac{e}{c}\vec{u}\times(\vec{\nabla}\times\vec{A})} - e\frac{\delta \phi}{\delta q_i}$$

$$u^k = \begin{pmatrix} c\gamma \\ \vec{v}\gamma \end{pmatrix}$$

$\vec{u}$ with upper index

$$\Rightarrow \frac{d\vec{p}}{dt} = +e\vec{E} + \frac{e}{c}\vec{u}\times\vec{B}$$

Lorentz force

Canonical momentum

$$p_\mu = -\frac{\delta L}{\delta u^\mu} = \underbrace{m\gamma u_\mu}_{\text{Kinematical momentum}} + \frac{e}{c}A_\mu$$

Electromagnetic field tensor

$$\vec{E} = -\frac{1}{c}\partial_t \vec{A} - \vec{\nabla}\phi$$

$$E_K = -\frac{\partial A_K}{\partial x_0} - \partial x_K A^0$$

$$= -\partial^0 A_K + \partial^K A^0$$

$$\vec{B} = \vec{\nabla} \times \vec{A} \qquad B_3 = \partial_1 A_2 - \partial_2 A_1$$

$$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha$$

em field tensor

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

dual field tensor

$$\tilde{F}_{\alpha\beta} = \frac{1}{2}\varepsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix}$$

$$\vec{E} \to \vec{B}$$
$$\vec{B} \to -\vec{E}$$

Transformation of $\vec{E}$ an $\vec{B}$

$F_{\alpha\beta}$ in K

$F'_{\alpha\beta}$ in K'

$$F^{\alpha\beta\,\prime} = L^{\alpha}{}_{\gamma} L^{\beta}{}_{\delta} F^{\gamma\delta}$$

$$F' = L F L^T$$

boost along 1 axis

$$L = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \quad E_1' = E_1 \qquad B_1' = B_1$$

$$E_2' = \gamma(E_2 - \beta E_3) \qquad B_2' = \gamma(B_2 + \beta E_3)$$

$$E_3' = \gamma(E_3 + \beta B_2) \qquad B_3' = \gamma(B_3 - \beta E_2)$$

General form

$$\vec{E}' = \gamma(\vec{E} + \vec{\beta} \times \vec{B}) - \frac{\gamma^2}{1+\gamma} \vec{\beta}(\vec{\beta} \cdot \vec{E})$$

$$\vec{B}' = \gamma(\vec{B} - \vec{\beta} \times \vec{E}) - \frac{\gamma^2}{1+\gamma} \vec{\beta}(\vec{\beta} \cdot \vec{B})$$

- E and B are mixed by Lorentz transformations

Covariant form of ME

tensors ∂_α, J^α, $F^{\alpha\beta}$, $\tilde{F}^{\alpha\beta}$

$\partial_\alpha F^{\alpha\beta} \sim J^\beta$

only possibility
by tensor structure
and dimensionality.

$$\partial_\alpha F^{\alpha\beta} = \frac{4\pi}{c} J^\beta$$

$\beta = 0$ $\quad \vec{\nabla}\cdot\vec{E} = 4\pi\rho$

$\beta = 1,2,3$ $\quad \vec{\nabla}\times\vec{B} - \frac{1}{c}\partial_t\vec{E} = \frac{4\pi}{c}\vec{J}$

Other covariant equation

$$\partial_\alpha \tilde{F}^{\alpha\beta} = 0$$

$\beta = 0$ $\quad \operatorname{div}\vec{B} = 0$

$\beta = 1\,2\,3$ $\quad \vec{\nabla}\times\vec{E} + \frac{1}{c}\partial_t\vec{B} = 0$

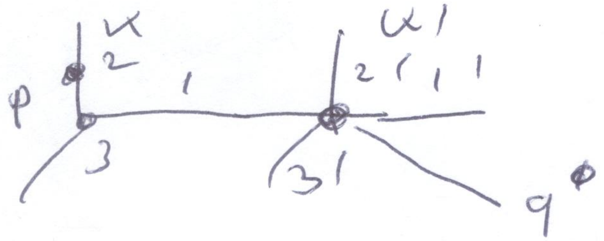
Bianchi identity

in terms of $F^{\alpha\beta}$

$$\partial^\alpha F^{\beta\gamma} + \partial^\beta F^{\gamma\alpha} + \partial^\gamma F^{\alpha\beta} = 0$$

Example

charge q at o in K'



$$E_1' = -\frac{q v t'}{r'^3} \qquad E_2' = \frac{q b}{r'^3} \qquad E_3' = 0$$

$$r' = \sqrt{b^2 + v^2 t'^2}$$

Field in K

$$E_1 = E_1' = -\frac{q \gamma v t}{\left(\sqrt{b^2 + \gamma^2 v^2 t^2}\right)^3}$$

$$t' = \gamma\left(t - \frac{v}{c^2} \underset{\substack{\parallel \\ 0 \text{ at } P}}{x_1}\right)$$

$$\Rightarrow t' = \gamma t$$

$$E_2 = \gamma E_2' = \frac{\gamma q b}{\left(\sqrt{b^2 + \gamma^2 v^2 t^2}\right)^3}$$

$$B_3 = \gamma \beta E_2' = \beta E_2$$

$$\beta \to 0 \qquad \vec{B} = \frac{q}{c} \frac{\vec{v} \times \vec{r}}{r^3}$$