Lecture #4

Gauss	$\vec{\nabla} E = 4\pi\rho$	
Poisson	$\vec{\nabla}^2 \phi = -4\pi\rho$	
Laplace	$\vec{\nabla}^2 \phi = 0$	

<u>Today</u>: Uniqueness

Solution of Laplace equation

Integral form of Gauss law

$$\int_V dV\, \vec{\nabla}\cdot\vec{E} = \int dV\, 4\pi\rho$$

$$\parallel$$

$$\int_{\partial V} da\, \vec{n}\cdot\vec{E}$$

e) <u>Poisson law</u>

$$\left.\begin{aligned} \vec{\nabla}\vec{E} &= 4\pi\rho \\ \vec{E} &= -\vec{\nabla}\phi \end{aligned}\right\} \Rightarrow \vec{\nabla}^2\phi = -4\pi\rho$$

Poisson equatia

$\rho=0 \quad \vec{\nabla}^2\phi = 0$ Laplace eq

solutions are called harmonic function.

f) <u>Uniqueness</u>

A vector field that satisfies

$$\vec{\nabla}\vec{E} = 0 \qquad \vec{\nabla}\times\vec{E}$$

and vanishes at least as $\frac{1}{r^2}$ for $r\to\infty$

vanishes everywhere

This follows from Liouville's theoreme:

if f is a harmonic function on $\mathbb{R}^n$

and f is bounded from above or below

then f is constant

<u>Reason</u> $\vec{E} = -\vec{\nabla}\phi$

$$\vec{\nabla}^2\phi = 0$$

$\vec{E} \sim \frac{1}{r^2}$ for $r \to \infty \Rightarrow \phi \sim \frac{1}{r}$ for $r \to \infty$

$\vec{\nabla}.\vec{E} = 0 \Rightarrow \exists$ no singularities

$\Rightarrow \phi$ is bounded $\Rightarrow \phi =$ constant

$\Rightarrow \vec{E} = 0$

Consequence: uniqueness theorem

$$\left.\begin{array}{l}\vec{\nabla}.\vec{E}_1 = 4\pi\rho_1 \\ \vec{\nabla}.\vec{E}_2 = 4\pi\rho_2\end{array}\right\} \Rightarrow \vec{\nabla}(\vec{E}_1 - E_2) = 0$$

statics $\Rightarrow \vec{\nabla}\times(\vec{E}_1 - \vec{E}_2) = 0$

$\Rightarrow E_1 - E_2 = 0$

Solutions of the Laplace eq

$$\vec{\nabla}^2\phi = 0$$

$z = r\cos\theta$
$x = r\sin\theta\cos\varphi$
$y = r\sin\theta\sin\varphi$

spherical coordinates

$$\vec{\nabla}^2\phi = \frac{1}{r}\partial_r^2(r\phi) + \frac{1}{r^2\sin\theta}\partial_\theta\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2}$$

$$\phi = \frac{u(r)}{r}P(\theta)\,Q(\varphi)$$

sepration of variables

$$\nabla^2\phi = \frac{PQ}{r}\partial_r^2 u + \frac{uQ}{rr^2\sin\theta}\partial_\theta(\sin\theta\,\partial_\theta P) \,\varphi$$

$$+ \frac{u\,P}{rr^2\sin^2\theta}\frac{\partial^2 Q}{\partial\varphi^2} = 0$$

divide by uPQ

$$\frac{u^{-1}}{r}\partial_r^2 u + \frac{P^{-1}}{rr^2\sin\theta}\partial_\theta(\sin\theta\,\partial_\theta P)$$

$$+ \frac{Q^{-1}}{rr^2\sin^2\theta}\frac{\partial^2 Q}{\partial\varphi^2} = 0$$

$\times r^3$

$$r^2 u^{-1}\partial_r^2 u + \frac{P^{-1}}{\sin\theta}\partial_\theta(\sin\theta\,\partial_\theta P)$$

$$+ \frac{Q^{-1}}{\sin^2\theta}\frac{\partial^2 Q}{\partial\varphi^2} = 0$$

only depends on r

only depends on θ, φ

$\Rightarrow$ both should be constant because the equation is valid for all r, θ, φ

$$r^2 u^{-1}\partial_r^2 u = c \Rightarrow \partial_r^2 u = c\frac{u}{r^2}$$

$$u = \alpha r^{\ell+1} \Rightarrow \ell(\ell+1) r^{\ell-2} = c\, r^{\ell-2}$$

$$\Rightarrow c = \ell(\ell+1)$$

$\frac{r_0}{V}$

then

$$\ell(\ell+1) \sin^2\theta + P^{-1} \sin\theta\, \partial_\theta (\sin\theta\, \partial_\theta P)$$

$$+ Q^{-1} \frac{\partial Q^2}{\partial \varphi^2} = 0$$

$$\Rightarrow \ell(\ell+1) \sin^2\theta + \sin\theta\, P^{-1} \partial_\theta (\sin\theta\, \partial_\theta P)$$

$$= -\left(\frac{\partial Q^2}{\partial \varphi^2}\right) Q^{-1}$$

function of θ function of φ

equal $\forall\ \theta, \varphi \Rightarrow$ they have to be constant

$$\Rightarrow Q^{-1} \frac{\partial^2 Q^2}{\partial \varphi^2} = -m^2 \Rightarrow Q = e^{\pm i m \varphi}$$

$$\Rightarrow \sin\theta\, \partial_\theta (\sin\theta\, \partial_\theta P) = \ell(\ell+1) \sin^2\theta\, P - m^2 P$$

This equation is known as the Legendre equation. $P = P(\cos\theta)$

The solutions are given by the associated Legendre polynomials

The combination PQ is given by $PQ = c' P_{\ell m}(\cos\theta)\, e^{im\varphi} \equiv \frac{1}{c'} Y_{\ell m}(\theta,\varphi)$

spherical harmonics

The constant c' is fixed by the normalization sum rule

$$\sum_{m=-\ell}^{\ell} Y_{\ell m}(\theta,\varphi)\, Y^*_{\ell m}(\theta,\varphi) = \frac{2\ell+1}{4\pi}$$

$$Y_{00} = \frac{1}{\sqrt{4\pi}}$$

$$Y_{11} = -\sqrt{\frac{3}{8\pi}} \sin\theta\, e^{i\phi} = -\sqrt{\frac{3}{8\pi}} \frac{x+iy}{r}$$

$$Y_{10} = \sqrt{\frac{3}{4\pi}} \cos\theta = \sqrt{\frac{3}{4\pi}} \frac{z}{r}$$

$$Y_{1-1} = \sqrt{\frac{3}{8\pi}} \sin\theta\, e^{-i\varphi} = \sqrt{\frac{3}{8\pi}} \frac{x-iy}{r}$$

We have found an infinite set of harmonic functions

$$\nabla^2 r^\ell\, Y_{\ell m}(\theta,\varphi) = 0$$

$$\ell(\ell+1) = (-\ell-1)((-\ell-1)+1)$$

$\Rightarrow u(r) = \frac{1}{r^\ell}$ is also solution

$$\Rightarrow \quad \vec{\nabla}^2 \frac{1}{r^{\ell+1}} Y_{\ell m}(\theta, \varphi) = 0$$

Most general solution of the Laplace eq.

$$\phi(r, \theta, \varphi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left(A_{\ell m} r^{\ell} + B_{\ell m} r^{-\ell-1} \right) Y_{\ell m}(\theta, \varphi)$$

This is a complete set of functions

Strategy for solution of Poisson eq

$$\nabla^2 \phi = -4\pi\rho$$

then $\nabla^2(\phi + \phi_h) = -4\pi\rho$

special solution

$$\vec{\nabla}^2 \phi_h = 0$$

Generally we have a charge distribution and boundary conditions

The ϕ_h are determined by the boundary conditions.

outside charge distribution ; $\rho = 0$

$\Rightarrow \vec{\nabla}^2 \phi = 0$ $\quad \phi \to 0$ for $r \to \infty \Rightarrow A_{\ell m} = 0$

inside charge distribution : solution should be regular $\Rightarrow B_{\ell m} = 0$

Lecture # 5

$$\vec{\nabla}^2 \phi = 0$$

$$\vec{\nabla}^2\phi = \frac{1}{r}\partial_r^2 (r\phi) + \frac{1}{r^2 \sin\theta}\partial_\theta \sin\theta \frac{\partial\phi}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2}$$

$$\phi = \frac{u(r)}{r} P(\theta)\, Q(\varphi)$$

$$u(r) = \begin{cases} r^{\ell+1} \\ \frac{1}{r^\ell} \end{cases} \qquad Q(\varphi) = e^{im\varphi}$$

$$+\sin\theta\,\partial_\theta(\sin\theta\,\partial_\theta P) + \ell(\ell+1)\sin^2\theta\, P - m^2 P = 0$$

$P = P(\cos\theta)$ is equation for Legendre polynomials

$$P = \cos\theta \Rightarrow \sin\theta\,\partial_\theta(\sin\theta\,\partial_\theta P) = -2\sin^2\theta\cos\theta$$

$\Rightarrow$ solution for $\ell = 1$ and $m = 0$.

Today: – Uniqueness
– Examples.

Some properties of spherical harmonics

$$Y_{\ell m}(\theta,\varphi) = \sqrt{\frac{2\ell+1}{4\pi}}\sqrt{\frac{(\ell-m)!}{(\ell+m)!}}\, P_{\ell m}(\cos\theta)\, e^{im\varphi}$$

$$r^2 \vec{\nabla}^2 Y_{\ell m}(\theta,\varphi) = -\ell(\ell+1)\, Y_{\ell m}(\theta,\varphi)$$

$$Y_{\ell -m}(\theta,\varphi) = (-1)^m\, Y^*_{\ell m}(\theta,\varphi)$$

orthogonality

$$\int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta\, d\theta\; Y^*_{\ell' m'}(\theta,\varphi)\, Y_{\ell m}(\theta,\varphi) = \delta_{\ell\ell'}\delta_{mm'}$$

completeness

$$\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} Y^*_{\ell m}(\theta',\varphi')\, Y_{\ell m}(\theta,\varphi) = \delta(\varphi-\varphi')\, \delta(\cos\theta - \cos\theta')$$

$\Rightarrow$ An arbitrary function of θ and φ can be expressed as

$$f(\theta,\varphi) = \sum a_{\ell m}\, Y_{\ell m}(\theta,\varphi)$$

sum rule

$$\sum_{m=-\ell}^{\ell} Y_{\ell m}(\theta,\varphi)\, Y^*_{\ell m}(\theta,\varphi) = \frac{2\ell+1}{4\pi}$$

$r < a \qquad \phi = \sum A_{l0}\, r^{l}\, Y_{l0}(\theta, \varphi)$

$r > a \qquad \phi = \sum B_{l0} \frac{1}{r^{l+1}}\, Y_{l0}(\theta, \varphi)$

$$Y_{l0}(0, \varphi) = \sqrt{\frac{2l+1}{4\pi}}$$

$r < a$ on z axis $\qquad \phi(z) = \frac{q}{a - z} = \sum_{n=0}^{\infty} \frac{q}{a}\left(\frac{z}{a}\right)^{n}$

$$\Rightarrow \quad A_{l0} = \frac{q}{a^{l+1}} \sqrt{\frac{4\pi}{2l+1}}$$

$r > a$ on z axis $\qquad \phi_z = \frac{q}{z-a} = q \sum_{n=0}^{\infty} \frac{a^{n}}{z^{n+1}}$

$$\Rightarrow \quad B_{l0} = q\, a^{l} \sqrt{\frac{4\pi}{2l+1}}$$

potential in entire space is then given by

$$\phi(r < a) = \sum_{l=0}^{\infty} \frac{q\, r^{l}}{a^{l+1}} \sqrt{\frac{4\pi}{2l+1}}\, Y_{l0}(\theta, \varphi)$$

$$\phi(r > a) = \sum_{l=0}^{\infty} \frac{q\, a^{l}}{r^{l+1}} \sqrt{\frac{4\pi}{2l+1}}\, Y_{l0}(\theta, \varphi)$$

do not depend on φ

Example 1

charge q at $r=0$

$$\rho(r) = q\,\delta^3(r)$$

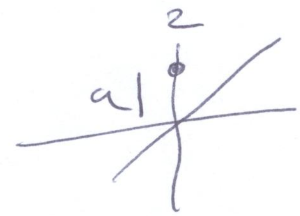
potential is spherically symmetric

$\Rightarrow$ only $l=0$ and $m=0$ are allowed

$$\Rightarrow \phi = A_{00} + \frac{B_{00}}{r}$$

$\phi \to 0$ for $r \to \infty \Rightarrow A_{00} = 0$

Example 2 charge q at $z=a$



$$\phi = \sum \left(A_{lm} r^l + \frac{B_{lm}}{r^{l+1}} \right) Y_{lm}(\theta,\varphi)$$

axial symmetry $\Rightarrow m=0$

We split the space in $r<a$ and $r>a$

$r<a$; potential is finite $\Rightarrow B_{lm} = 0$

$r>a$; potential vanishes for $r \to \infty$

$\Rightarrow A_{lm} = 0$

We are going to determine the coëfficients from the potential on the z axis where it is given by $\phi(z) = \dfrac{q}{|z-a|}$

$r < a \qquad \phi = \sum A_{\ell 0}\, r^{\ell}\, Y_{\ell 0}(\theta, \varphi)$

$r > a \qquad \phi = \sum B_{\ell 0} \frac{1}{r^{\ell+1}} Y_{\ell 0}(\theta, \varphi)$

$$Y_{\ell 0}(0, \varphi) = \sqrt{\frac{2\ell+1}{4\pi}}$$

$r < a$ on z axis $\qquad \phi(z) = \frac{q}{a - z} = \sum_{n=0}^{\infty} \frac{q}{a}\left(\frac{z}{a}\right)^n$

$$\Rightarrow \quad A_{\ell 0} = \frac{q}{a^{\ell+1}} \sqrt{\frac{4\pi}{2\ell+1}}$$

$r > a$ on z axis $\qquad \phi_z = \frac{q}{z - a} = q \sum_{n=0}^{\infty} \frac{a^n}{z^{n+1}}$

$$\Rightarrow \quad B_{\ell 0} = q\, a^{\ell} \sqrt{\frac{4\pi}{2\ell+1}}$$

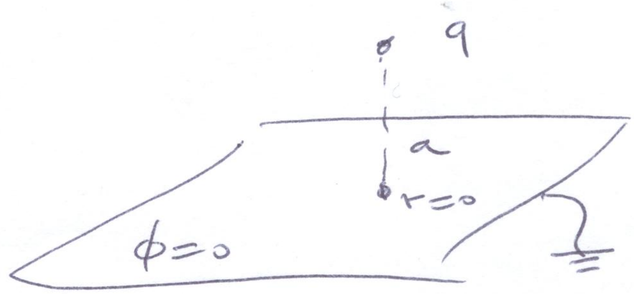
potential in entire space is then given by

$$\phi(r < a) = \sum_{\ell=0}^{\infty} \frac{q\, r^{\ell}}{a^{\ell+1}} \sqrt{\frac{4\pi}{2\ell+1}}\, Y_{\ell 0}(\theta, \varphi)$$

$$\phi(r > a) = \sum_{\ell=0}^{\infty} \frac{q\, a^{\ell}}{r^{\ell+1}} \sqrt{\frac{4\pi}{2\ell+1}}\, Y_{\ell 0}(\theta, \varphi)$$

do not depend on φ

Example 3



ϕ is axially symmetric $\Rightarrow m=0$

again split space in $r<a$ and $r>a$

$r<a$ potential is finite $\Rightarrow B_{\ell m}=0$

$$\phi=\sum_{\ell=0}^{\infty} A_{\ell 0}\, r^{\ell}\, Y_{\ell 0}(\theta,\varphi)$$

$r>a$ $\quad \phi\to 0$ for $r\to\infty \Rightarrow A_{\ell m}=0$

$$\Rightarrow \phi=\sum_{\ell=0}^{\infty} B_{\ell 0}\frac{1}{r^{\ell+1}} Y_{\ell 0}(\theta,\varphi)$$

Potential vanishes on the plane

so $\phi(\theta=\frac{\pi}{2})=0$

$$Y_{\ell 0}(\tfrac{\pi}{2},\varphi)=(1+(-1)^{\ell})(-1)^{\ell/2}c_{\ell}$$

$\Rightarrow$ ℓ must be odd

$$\Rightarrow r<a \qquad \phi=\sum_{\ell=0}^{\infty} A_{\ell 0}(1-(-1)^{\ell})\, r^{\ell}\, Y_{\ell 0}(\theta,\varphi)$$

on z axis $\quad z\to a$ then $\phi\to\frac{q}{a-z}=\sum_n \frac{q z^n}{a^{n+1}}$

$$\Rightarrow A_{\ell 0}=\frac{q}{a^{\ell+1}}\sqrt{\frac{4\pi}{2\ell+1}} \qquad \left(Y_{\ell 0}(\theta=0,\varphi)=\sqrt{\frac{2\ell+1}{4\pi}}\right)$$

$$\Rightarrow r<a \quad \phi(\theta,\varphi) = \sum_{\ell=0}^{\infty} \frac{q}{a^{\ell+1}} r^\ell \sqrt{\frac{4\pi}{2\ell+1}} (1-(-1)^\ell) Y_{\ell 0}(\theta,\varphi)$$

$r>a$ on z axis $z\to a$ $\quad \phi \to \frac{q}{z-a} = \sum_n \frac{q\,a^n}{z^{n+1}}$

$$\Rightarrow A_{\ell 0} = q\, a^\ell \sqrt{\frac{4\pi}{2\ell+1}}$$

$$\phi(\theta,\varphi) = \sum_{\ell=0}^{\infty} \frac{q\,a^\ell}{r^{\ell+1}} \sqrt{\frac{4\pi}{2\ell+1}} (1-(-1)^\ell) Y_{\ell 0}(\theta,\varphi)$$

The first term is exactly the potential of a charge at $r=a$.

What about the $(-1)^\ell$ term

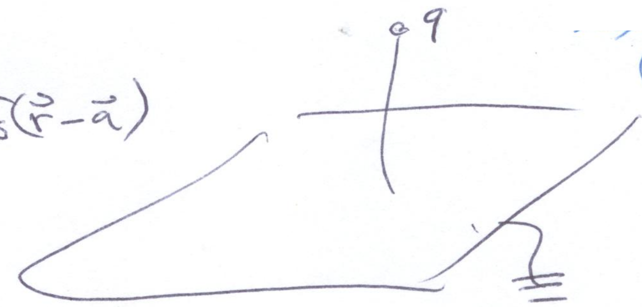
$$(-1)^\ell Y_{\ell 0}(\theta,\varphi) = Y_{\ell 0}(\pi-\theta,\varphi)$$

$$r<a \quad \phi(r) = \sum_{\ell=0}^{\infty} \frac{(-q)}{a^{\ell+1}} r^\ell \sqrt{\frac{4\pi}{2\ell+1}}\, Y_{\ell 0}(\pi-\theta,\varphi)$$

$$r>a \quad \phi(r) = \sum_{\ell=0}^{\infty} \frac{(-q)}{r^{\ell+1}} a^\ell \sqrt{\frac{4\pi}{2\ell+1}}\, Y_{\ell 0}(\pi-\theta,\varphi)$$

potential of charge $-q$ at $z=-a$

Alternate Method



$$\nabla^2(\phi_q + \phi_H) = -4\pi\,\delta(\vec r - \vec a)$$

$$\phi_q = \frac{q}{|\vec r - \vec a|}$$

$$\vec\nabla^2 \phi_H = 0 \quad \Rightarrow \quad \phi_H = \sum_l \left(A_{l0}\, r^l + \frac{B_{l0}}{r^{l+1}}\right) Y_{l0}$$

$$\phi_q \underset{r<a}{=} q \sum a^{-l-1} r^l \sqrt{\frac{4\pi}{2l+1}}\, Y_{l0}$$

$$\underset{r>a}{=} q \sum r^{-l-1} a^l \sqrt{\frac{4\pi}{2l+1}}\, Y_{l0}$$

$\phi_q + \phi_H = 0$ on $z=0$ plane ie for $\theta = \frac{\pi}{2}$

$$Y_{l0}\left(\theta = \tfrac{\pi}{2}, \varphi\right) = \left(1 + (-1)^l\right) c_l$$

Vanishes for odd l.

$\Rightarrow$ A_{l0} and B_{l0} for odd l are not determined by the boundary conditions

$r<a$ $\quad A_{l0} = -q\,\frac{1}{a^{l+1}}\sqrt{\frac{4\pi}{2l+1}}$

l is even

on z axis is $\quad \sum_{l=0}^{\infty} -q\,\frac{r^{2l}}{a^{2l+1}}\sqrt{\frac{4\pi}{4l+1}}\sqrt{\frac{4l+1}{4\pi}} = \frac{-q}{a\left(1 - \frac{r^2}{a^2}\right)}$

$$= \frac{-qa}{a^2 - r^2} = \frac{-qa}{2a}\left(\frac{1}{a-r} + \frac{1}{a+r}\right)$$

There can be no singularity at $r = a$

This has to be cancelled by the odd l.

$$\sum_{l=0}^{\infty} A_{2l+1\,0}\, r^{2l+1} \sqrt{\frac{2(2l+1)+1}{4\pi}} \quad \Bigg| \quad A_{2l+1\,0} = \sqrt{\frac{4\pi}{2(2l+1)}}\,\frac{1}{a^{2l+2}}\,q$$

$$\sum_{l=0}^{\infty} \frac{q\, r^{2l+1}}{a^{2l+2}} = \frac{qr}{a^2}\frac{1}{1 - \frac{r^2}{a^2}} = \frac{qr}{a^2 - r^2}$$

combining even and odd givs

$$A_{\ell 0} = -q \frac{(-1)^\ell}{a^{\ell+1}} \sqrt{\frac{4\pi}{2\ell+1}}$$

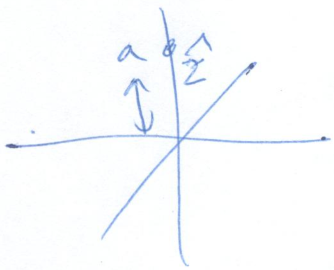
on the z axis this give $\frac{q(r-a)}{a^2-r^2} = \frac{-q}{a+r}$

this is the potential of charge $-q$ at $z=a$

$$\Rightarrow \quad \phi^H_{r<a} = \sum_{\ell=0}^{\infty} (-q) \frac{(-1)^\ell}{a^{\ell+1}} r^\ell \sqrt{\frac{4\pi}{2\ell+1}} Y_{\ell 0}(\theta, \varphi)$$

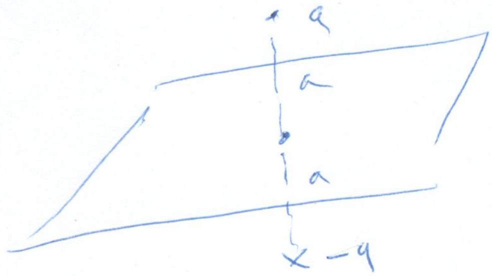
The analysis for $r>a$ is completely analogous.

Lecture #6



$$\phi(r<a) = \sum_{\ell=0}^{\infty} \frac{q\, r^{\ell}}{a^{\ell+1}} \sqrt{\ldots}$$

$$\phi(r>a) = \sum_{\ell=0}^{\infty} \frac{q\, a^{\ell}}{r^{\ell+1}}$$



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Spherical har

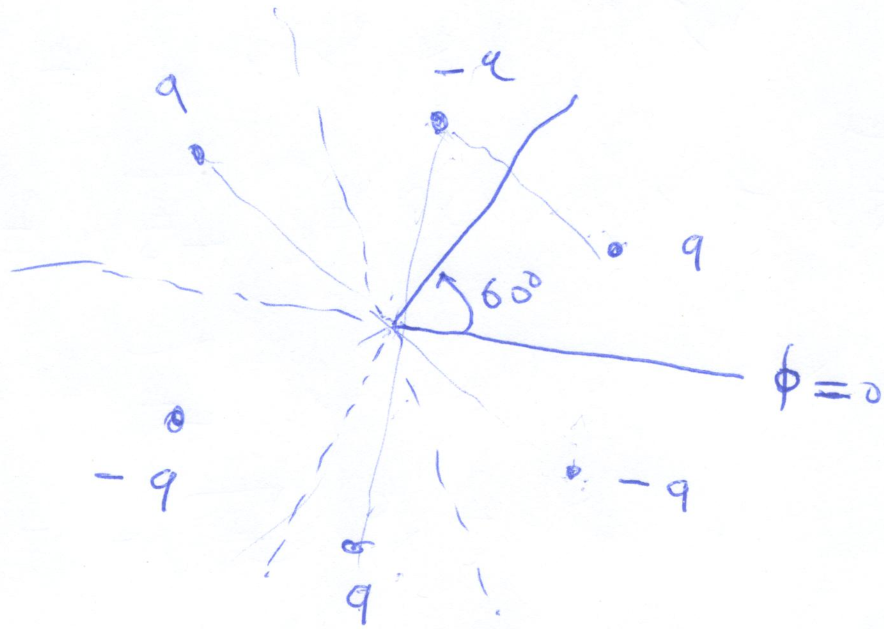
Today: - Uniqueness theore
- general proper
- image charges

This problem can be solved by image charges.
The trick is to find other charges that give the same boundary condition

$\Rightarrow$ Solution on the right of the plane is the same, not on the left

$$\phi(\vec{r}) = \frac{q}{|\vec{r}-\vec{a}|} - \frac{q}{|\vec{r}+\vec{a}|}$$

2nd Example of image charges



Expansion of $\frac{1}{|\vec r - \vec r'|}$ in spherical harmonics

choose $|\vec r| > |r'|$

$$\frac{1}{|\vec r - \vec r'|} = \sum \frac{1}{r^{\ell+1}} A_{\ell m} P_\ell Y_{\ell m}(\theta,\varphi)$$

choose $\vec r'$ on z axis

on z axis: $$\frac{1}{|r-r'|} = \sum_{k=0} \frac{r'^k}{r^{k+1}}$$

~~θ~~ no dependence on $\varphi \Rightarrow m=0$

$$\Rightarrow A_{\ell 0} = \frac{r'^\ell}{Y_{\ell 0}(0,\varphi)} = r'^\ell \left(\frac{4\pi}{2\ell+1}\right)^{1/2} Y_{\ell 0}(0,\varphi)$$

$$\frac{1}{|\vec r-\vec r'|} = \sum \frac{1}{r^{\ell+1}} r'^\ell \frac{4\pi}{2\ell+1} Y_{\ell 0}(0,\varphi)\, Y_{\ell 0}(\theta,\varphi)$$

$$Y_{\ell 0}(\theta',\varphi)\, Y_{\ell 0}(\theta,\varphi)$$

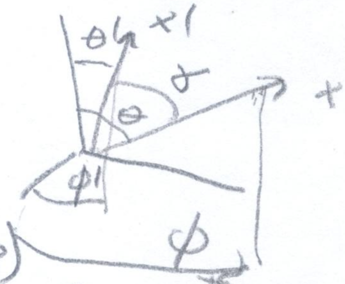
only depends on angle between $\vec r$ and $\vec r'$

$$Y_{\ell 0}(\theta,\varphi) = \sqrt{\frac{2\ell+1}{4\pi}}\, P_\ell(\cos\theta)$$

$$Y_{\ell 0}(0,0) = \sqrt{\frac{2\ell+1}{4\pi}}$$

addition theorem

$$P_\ell(\cos\gamma) = \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{\ell} Y^*_{\ell m}(\theta',\varphi') \times Y_{\ell m}(\theta,\varphi)$$

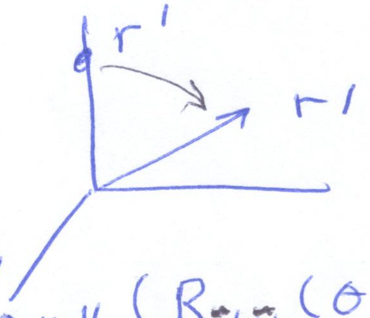


$$\Rightarrow \frac{1}{|\vec r-\vec r'|} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{4\pi}{(2\ell+1)} \frac{r'^\ell}{r^{\ell+1}} Y^*_{\ell m}(\theta',\varphi')\, Y_{\ell m}(\theta,\varphi)$$

$$\frac{1}{|\vec{r}-\vec{r}'|} = \sum_\ell \frac{r'^\ell}{r^{\ell+1}} \sqrt{\frac{4\pi}{2\ell+1}}\, Y_{\ell 0}(\theta,\varphi)$$

on z-axis

now rotate $r' \to (\theta', \varphi')$



then $Y_{\ell m}(\vec{r}) = \sum_{m''} \mathcal{D}^\ell_{m'' m}\, Y_{\ell m''}\big(\underbrace{R_{\theta'\varphi'}(\theta,\varphi)}_{\text{new } \theta, \varphi \text{ coordinates of } \vec{r}}\big)$

$m \to 0$, $m \to 0$

Wigner $\mathcal{D}$ matrices

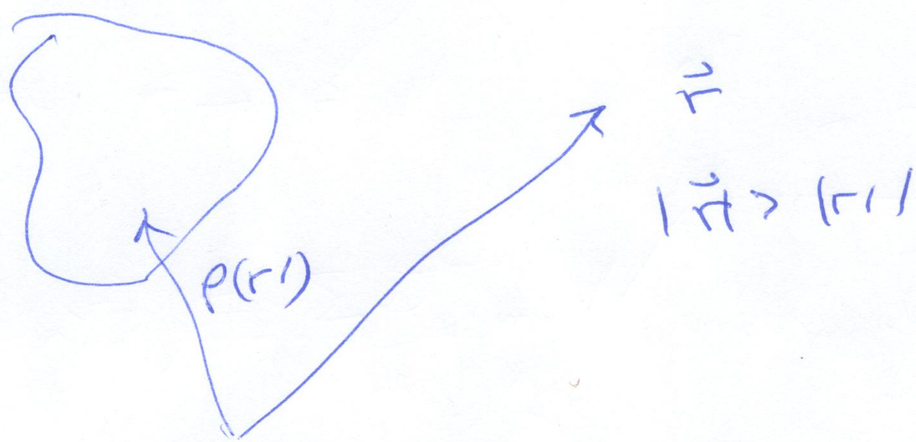
$$\mathcal{D}^\ell_{m''0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y^*_{\ell m''}(\theta', \varphi')$$

$$\Rightarrow \frac{1}{|\vec{r}-\vec{r}'|} = \sum_{\ell, m''} \frac{r'^\ell}{r^{\ell+1}} \frac{4\pi}{2\ell+1} Y^*_{\ell m''}(\theta', \varphi')\, Y_{\ell m''}(\theta, \varphi)$$

note that $r > r'$

Multipole expansion

a)



$$\phi(r'') = \int d^3r' \, \rho(r') \frac{1}{|\vec{r}\,' - \vec{r}\,|}$$

$$= \int d^3r' \, \rho(r') \sum_{lm} \frac{r'^l}{r^{l+1}} \frac{4\pi}{2l+1} \rho(r') Y^*_{lm}(\theta', \varphi') \times Y_{lm}(\theta, \varphi)$$

multipole moment

$$Q_{lm} = \int d^3r' \, r'^l \rho(\vec{r}\,') \, Y^*_{lm}(\theta' \varphi')$$

$$\Rightarrow \phi(r) = \sum_{l=0}^{\infty} \frac{4\pi}{2l+1} \frac{1}{r^{l+1}} \sum_{m=-l}^{l} Y_{lm}(\theta, \varphi) \, Q_{lm}$$

$l=0$ monopole

$l=1$ dipole

$l=2$ quadrapole

Lecture #7

$$\frac{1}{|\vec{r}-\vec{r}'|} = \sum_{\ell m} \frac{r'^{\ell}}{r^{\ell+1}} \frac{4\pi}{2\ell+1} Y^*_{\ell m}(\theta',\varphi) Y_{\ell m}(\theta,\varphi)$$

$r > r'$

$r < r'$ then $r' \leftrightarrow r$

multipole expansion

$$Q_{\ell m} = \int d^3r'\, r'^{\ell} \rho(r') Y^*_{\ell m}(\theta',\varphi)$$

$$\phi(r) = \sum_{\ell=0}^{\infty} \frac{4\pi}{2\ell+1} \frac{1}{r^{\ell+1}} \sum_{m=-\ell}^{\ell} Y_{\ell m}(\theta,\varphi) Q_{\ell m}$$

$\ell=0$ monopole
$\ell=1$ dipole
$\ell=2$ quadrupole

Today - image charges
- Dipole moment
- Electrostatics of conductors

2d Gauss theorem $\int_D \vec{\nabla}\cdot\vec{A}\, d^2a = \int_{\partial D} \vec{A}\cdot\hat{n}\, ds$

Stokes theorem $\int_D \vec{\nabla}\times\vec{A}\, d^2a = \int_{\partial D} \vec{A}\cdot d\vec{s}$

2d E field $\vec{E} = q\frac{\vec{x}}{r^2} + q\frac{\vec{y}}{r^2}$

$$\int_D \vec{\nabla}\cdot\vec{E} = \int_{\partial D} q\left(\frac{\vec{x}\cdot\hat{x}}{r^2} + \frac{\vec{y}\cdot\hat{y}}{r^2}\right) ds = \frac{q}{r} 2\pi r = 2\pi q$$

b) <u>Dipole moment</u>

$$\vec{p} = \int d^3x' \, \rho(x') \, \vec{x}'$$

$$Q_{10} = \sqrt{\frac{3}{4\pi}} \int z' \rho(x') \, d^3x' = \sqrt{\frac{3}{4\pi}} \, p_z$$

$$Q_{11} = -\sqrt{\frac{3}{8\pi}} \int (x' - iy') \, \rho(x') \, d^3x'$$

$$= -\sqrt{\frac{3}{8\pi}} \, (p_x - i p_y)$$

$$Q_{1-1} = - Q_{11}^* = \sqrt{\frac{3}{8\pi}} \, (p_x + i p_y)$$

The potential of a dipole is given by

$$\phi(r) = \frac{4\pi}{\underset{\nearrow}{3}} \frac{1}{r^2} \Big(Y_{11} \Big(-\sqrt{\frac{3}{8\pi}}\Big)(p_x - i p_y)$$

$$\quad 2l+1 \qquad\qquad + Y_{10} \sqrt{\frac{3}{4\pi}} \, p_z$$

$$+ Y_{1-1} \sqrt{\frac{3}{8\pi}} \, (p_x + i p_y)$$

$$Y_{11} = -\sqrt{\frac{3}{8\pi}} \, (x + iy)/r \qquad Y_{10} = \sqrt{\frac{3}{4\pi}} \, \frac{z}{r}$$

$$Y_{1-1} = \sqrt{\frac{3}{8\pi}} \, \frac{(x - iy)}{r}$$

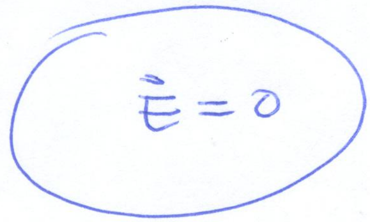
$$\phi(r) = \frac{4\pi}{3} \frac{1}{r^3} \Big((x+iy)(p_x - i p_y) \frac{3}{8\pi} + z \, p_z \frac{3}{4\pi}$$

$$+ (x - iy)(p_x + i p_y) \frac{3}{8\pi}$$

$$= \frac{1}{r^3} \left(x p_x + y p_y + z p_z \right) = \frac{1}{r^3} (\vec{r} \cdot \vec{p})$$

Electrostatics of conductors

a)



$\vec{E} = 0 \Rightarrow \vec{\nabla}\vec{E} = 0 \Rightarrow \rho = 0$

$\Rightarrow$ we can only have a surface charge density on a conductor.

$\vec{E} = 0 \quad \vec{E} = \vec{\nabla}\phi \Rightarrow$

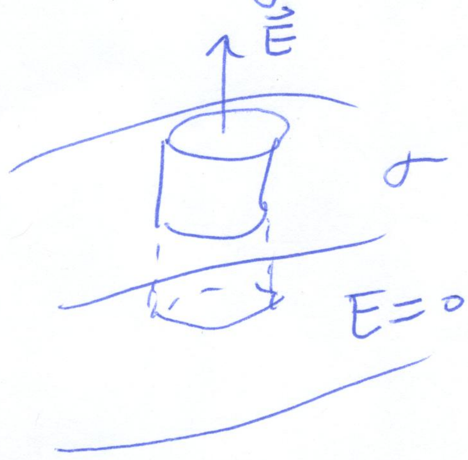
- conductor is an equipotential surface

$$\phi_{12} = -\int_1^2 \vec{E} \cdot ds$$

- 

$\vec{E} \perp$ surface, otherwise the electrons will rearrange

- Surface charge density



$$\oint \vec{E} \cdot \vec{n}\, da = 4\pi\sigma A$$

$$\Rightarrow E_\perp A = 4\pi\sigma A$$

$$E_\perp = 4\pi\sigma$$

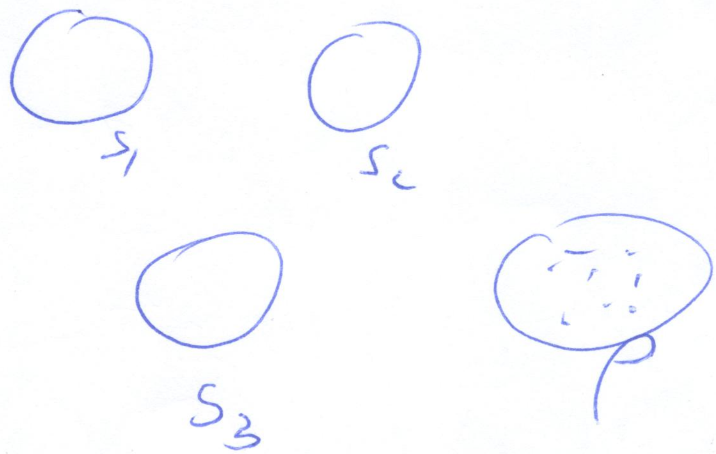
Lecture # 8

Today: Uniqueness Theorem
Green's Function
Application

Uniqueness theorem

give ρ, conducting surfaces S_i with charge Q_i or potential ϕ_i, then the electric field is determined uniquely

Proof



Suppose we have two different potentials

$$\vec{E}_1 = -\vec{\nabla}\Psi_1 \qquad E_2 = -\vec{\nabla}\Psi_2$$

then $$I = \int_{V = \mathbb{R}^3/\cup(S_i)} d^3r \, (\vec{\nabla}\Psi_1 - \vec{\nabla}\Psi_2)^2 \neq 0$$

apply Gauss theorem to $\vec{\nabla}[(\Psi_1^i - \Psi_2^i)\vec{\nabla}(\Psi_1^i - \Psi_2^i)]$

$$= \int_V d^3r \; \vec{\nabla}(\Psi_1 - \Psi_2)\,\vec{\nabla}(\Psi_1 - \Psi_2)$$

$$= \sum_i \int_{S_i} da \; \vec{n}\,(\Psi_1^i - \Psi_2^i)\,\vec{\nabla}(\Psi_1^i - \Psi_2^i) - \int_V d^3r (\Psi_1 - \Psi_2) \times \underbrace{\vec{\nabla}^2(\Psi_1 - \Psi_2)}_{=0}$$

Ψ_k^i is constant on S_i

$$\vec{\nabla}^2\Psi_1 = -4\pi\rho \qquad \vec{\nabla}^2\Psi_2 = -4\pi\rho$$

$$= \sum_i \int_{S_i} (\Psi_1^i - \Psi_2^i)\,\vec{n}\,.(\vec{E}_{1i} - \vec{E}_{2i})$$

$$= \sum_i (\Psi_1^i - \Psi_2^i)(Q_1^i - Q_2^i) \Rightarrow \text{qed}$$

Green's function

Definition $\vec{\nabla}^2_{x'} G(\vec{x},\vec{x}') = -4\pi\,\delta(\vec{x}-\vec{x}')$

So $G(\vec{x},\vec{x}') = \dfrac{1}{|\vec{x}-\vec{x}'|} + F(\vec{x},\vec{x}')$

with $\vec{\nabla}^2_{x'} F(\vec{x},\vec{x}') = 0$

F is determined by the boundary conditions

for a single charge in vacuum

$$G(\vec{x},\vec{x}') = \frac{1}{|\vec{x}-\vec{x}'|}$$

is potential at x from unit charge at x'

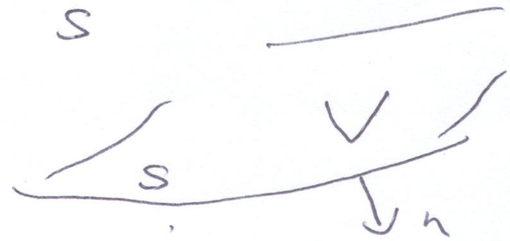
Green's function is symmetric

$$\int d^3r\, G(\vec{r},\vec{r}_1)\underbrace{\vec{\nabla}^2_r G(\vec{r},\vec{r}_2)}_{-4\pi\delta(\vec{r}-\vec{r}_2)} = \int d^3r\, \underbrace{\vec{\nabla}^2_r G(\vec{r},\vec{r}_1)}_{-4\pi\delta(\vec{r}-\vec{r}_1)} G(\vec{r},\vec{r}_2)$$

$$\Rightarrow -4\pi\, G(\vec{r}_2,\vec{r}_1) = -4\pi\, G(\vec{r}_1,\vec{r}_2)$$

Application of Green's Function

Gauss: $$\int_V \vec{\nabla}\cdot\vec{A}\, d^3x = \oint_S \vec{A}\cdot\vec{n}\, da$$



choose $\vec{A} = \phi\, \vec{\nabla}\psi$

arbitrary

Then $$\vec{\nabla}\vec{A} = \phi\, \vec{\nabla}^2\psi + \vec{\nabla}\phi\, \vec{\nabla}\psi$$

$$\vec{A}\cdot\vec{n} = \phi\, \vec{\nabla}\psi\, \vec{n} \equiv \phi\, \partial_n\psi$$

$$\Rightarrow \int_V (\phi\, \vec{\nabla}^2\psi + \vec{\nabla}\phi\, \vec{\nabla}\psi)\, d^3x = \int_S \phi\, \vec{\nabla}\psi\cdot\vec{n}\, da$$

Green I

subtract same eq with ϕ and ψ interchanged

Green II: $$\int_V (\phi\, \vec{\nabla}^2\psi - \psi\, \vec{\nabla}^2\phi)\, d^3x' = \oint_S \left(\phi \frac{\partial\psi}{\partial n} - \psi \frac{\partial\phi}{\partial n}\right) da'$$

choose ϕ a potential $\vec{\nabla}^2\phi = -4\pi\rho$

and ψ a Green's function $\psi = G(x', x)$

$$\nabla^2_{x'}\, G(\vec{x}', \vec{x}) = -4\pi\, \delta^3(x', x)$$

$$-4\pi\phi(x) + 4\pi \int \rho(x')\, G(x', x)\, d^3x'$$

$$= \oint_S \phi(x') \left(\vec{\nabla}_{x'} G(x', x) - G(x', x)\, \vec{\nabla}\phi\right)\cdot\vec{n}\, da$$

choose Green's function such that $G(x', x) = 0$ if $x' \in S$

$$\Rightarrow \phi(x) = \int \rho(x')\, G(x', x)\, d^3x' - \oint_S \phi(x')\, \vec{\nabla}_{x'} G(x', x)\cdot\vec{n}\, da$$

Lecture # 9

Green's Function $\vec{\nabla}^2_{x'} G(\vec{x}, x') = -4\pi \delta^3(x - x')$

$$G(x, y) = G(y, x)$$

$$\phi(x) = \int \rho(x') G(x', x) d^3x - \oint_S \phi(x') \vec{\nabla}_{x'} G(x', x) \cdot \hat{n} da$$

Today
- Energy
- stress tensor
- Example

4. Energy and stress in an electstatic Field

1) Energy

Work done to bring a small charge dq_i from ∞ to r_i

$$\delta W_i = (\phi(r_i) - \phi(\infty))\,\delta q_i$$
$$= -\int_{\infty}^{r_i} \vec{E}\cdot d\vec{s}\;\delta q_i$$

total work for many charges broaght from infinity

$$\delta W = \sum_i \delta W_i = \sum_i \phi(r_i)\,\delta q_i$$
$$= \int d^3r\;\phi(r)\,\delta\rho(r)$$

$$\vec{\nabla}\vec{E} = -4\pi\rho \qquad \Rightarrow \qquad \delta\rho = \frac{1}{4\pi}\vec{\nabla}\delta\vec{E}$$

$$\Rightarrow \delta W = \int d^3r\;\phi(r)\,\frac{1}{4\pi}\vec{\nabla}\delta E$$
$$\underset{\text{partial integration}}{=} -\int d^3r\;\vec{\nabla}\phi\;\frac{1}{4\pi}\delta E$$

we integrate over all space $\Rightarrow$ surface term vanishes

$$\Rightarrow \delta W = \frac{1}{4\pi}\int d^3r\;\vec{E}\cdot\delta E$$
$$= \frac{1}{8\pi}\int d^3r\;\delta\vec{E}^2$$

Work done to change the Field strength from $E_0 \to E_F$

$$W = \frac{1}{8\pi}\int d^3r\,(E_f^2 - E_0^2)$$

$\vec{E}$ is changed by taking charges from ∞ to r_i

$\Rightarrow$ energy density is $\dfrac{\vec{E}^2}{8\pi}$

b) <u>Stress tensor</u>

Force on charges inside S

$$F_i = \int_V d^3r \, \rho(\vec{r}) \, E_i(\vec{r})$$

$\leftarrow$ i'th component of $\vec{E}$ $\quad$ $i=1,2,3$ $\quad$ $\partial V = S$

$$= \frac{1}{4\pi} \int_V d^3r \, \vec{\nabla}\cdot\vec{E} \, E_i$$

$$= \frac{1}{4\pi} \int_V d^3r \left(\partial_j (E_j E_i) - E_j \partial_j E_i \right)$$

$$\vec{\nabla}\times\vec{E} = 0 \Rightarrow \varepsilon_{ijk}(\partial_j E_k - \partial_k E_j) = 0$$

$$\Rightarrow \partial_j E_k - \partial_k E_j = 0 \quad k \neq j$$

and trivially $= 0$ for $k = j$

$$\Rightarrow E_j \partial_j E_i = E_j \partial_i E_j + \underbrace{E_j \partial_j E_i - E_j \partial_i E_j}_{=0}$$

$$= \frac{1}{2} \partial_i E_j^2$$

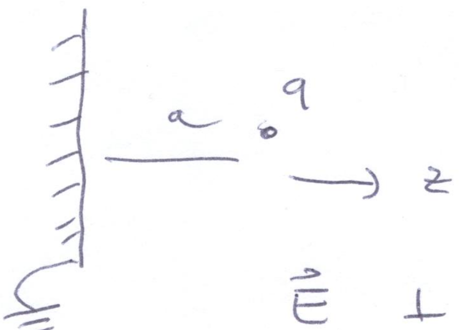
$$\Rightarrow F_i = \frac{1}{4\pi} \int_V d^3r \, \partial_j \left(E_j E_i - \delta_{ij} \frac{\vec{E}^2}{2} \right)$$

$$= \int_S ds_j \, T_{ji}$$

Gauss

Maxwell stress tensor $\quad T_{ji} = \frac{1}{4\pi}\left(E_j E_i - \delta_{ij} \frac{\vec{E}^2}{2} \right)$

Example



$\vec{E} \perp$ surface at surface

$\Rightarrow \vec{E} = (0, 0, E_z)$

$$T_{ji}\Big|_s = \begin{pmatrix} -\frac{1}{2}E_z^2 & & \\ & -\frac{1}{2}E_z^2 & \\ & & \frac{1}{2}E_z^2 \end{pmatrix} \frac{1}{4\pi}$$

$$F_i = \int ds_i T_{ji}$$

$$= \int \hat{n}\, ds\, T_{33} = -\int_{ds} \frac{1}{8\pi} E_z^2$$

normal to the outside from the point of view of the charge

E is the field due to the charge and image charge

$$\vec{E} = -\frac{q}{(a^2+\rho^2)} 2\cos\varphi$$

$$= -\frac{2qa}{(a^2+\rho^2)^{3/2}} \quad \text{at } \varphi = 0$$

$$\vec{F} = -\int ds \frac{1}{8\pi} E_z^2 = -\int ds \frac{1}{8\pi} \frac{4q^2a^2}{(a^2+\rho^2)^3}$$

$$\int \rho\, d\rho\, d\varphi$$

$$= -\frac{2\pi}{8\pi} 4 \int_0^\infty \rho\, d\rho \frac{q^2a^2}{(a^2+\rho^2)^3} = \frac{q^2}{a^2} \int_0^\infty \frac{\rho\, d\rho}{(1+\rho^2)^3}$$

$$= -\frac{q^2}{4a^2}$$ correct

Energy in terms of charges

$$W_E = \frac{1}{8\pi} \int_V \vec{E}^2 d^3r$$

$$= \frac{1}{8\pi} \int_V \vec{\nabla}\phi \, \vec{\nabla}\phi \, d^3r$$

$$= -\frac{1}{8\pi} \int_V \phi \, \vec{\nabla}^2 \phi \, d^3r$$

$$= \frac{1}{2} \int d^3r \; \phi(r) \rho(r)$$

$$= \frac{1}{2} \int d^3r \, d^3r' \frac{\rho(r)\rho(r')}{|\vec{r} - \vec{r}'|}$$

↑

factor $\frac{1}{2}$ because

each pair of charges is counted only once
For a collection of point charge

$$W = \frac{1}{2} \sum_{i \neq j} \frac{q_i q_j}{|\vec{r}_i - \vec{r}_j|}$$

the energy is always because of the field from another charge.

we see later that the energy of a magnetic field is $W_B = \frac{1}{8\pi} \int d^3r \, \vec{B}^2$

$$T + W_E + W_B = \text{constant}$$

↑ Kinetic energy

Lecture #10

$$W = \frac{1}{8\pi} \int d^3r \, \vec{E}^2$$

Stress tensor

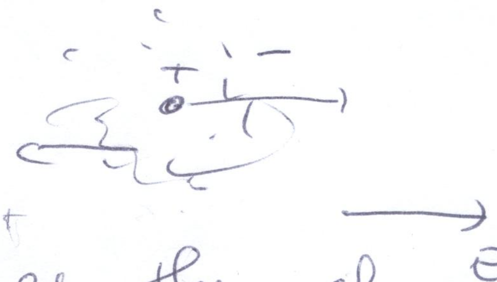
$$F_i = \int ds_i \, T_{ji}$$

$$T_{ji} = \frac{1}{4\pi} \left(E_j E_i - \frac{\vec{E}^2}{2} \delta_{ij} \right)$$

Today
- Electrostatics in matter
- boundary conditions

5) Electrostatics in matter

- in conductors electrons can move freely
- in dielectric the electrons are bound to the atoms



in electric field the charge distribution of the electrons gets displaced wrt the nucleus and the atom gets a dipole moment.

model This effect can be approximated by a dipole density $\vec{p}(r)$

This gives the potential

$$\phi(r) = \sum_i \frac{\vec{p}_i (\vec{r} - \vec{r}_i)}{|(\vec{r} - r_i)|^3}$$

$$= -\sum p_i \, \vec{\nabla}_{\vec{r}} \frac{1}{|\vec{r} - \vec{r}_i|}$$

$$= -\int d^3 r' \, \vec{p}(r') \, \underbrace{\vec{\nabla}_{\vec{r}}}_{= -\vec{\nabla}_{r'}} \frac{1}{|\vec{r} - \vec{r}'|}$$

$$= \int d^3 r' \left(\nabla_{r'} \left(p(r') \frac{1}{|\vec{r} - \vec{r}'|} \right) - \frac{(\nabla_{r'} p(r'))}{|r - r'|} \right)$$

$$= \int_{S=\partial V} \frac{\vec{P}(r') \cdot d\vec{a}}{|r-r'|} + \int d^3r' \, \frac{\left(-\vec{\nabla}_{\vec{r}'} P(\vec{r}')\right)}{|r-r'|}$$

potential of surface charge density on S

$\sigma = \vec{P} \cdot \vec{n}$

induced charge density

$\rho_b = -\vec{\nabla}_{r'} P(r')$

- if the dipole density is constant then $\vec{\nabla}_{r'} \vec{P}(r') = 0$ and we only have a surface charge density

effect of induced charge density

$$\vec{\nabla} \vec{E} = 4\pi(\rho + \rho_{ind})$$

$$= 4\pi(\rho - \vec{\nabla} \vec{P})$$

$$\Rightarrow \vec{\nabla}(\underbrace{\vec{E} + 4\pi\vec{P}}_{\equiv \vec{D}}) = 4\pi \rho$$

free charges

For linear media we have $\vec{D} = \varepsilon \vec{E}$

For a surface charge density we have:

$\sigma_{ind} = (\vec{P}\cdot\vec{n})$

$$\vec{E} = 4\pi(\sigma - \sigma_{ind})\cdot\vec{n}$$

$$= 4\pi(\sigma - (\vec{P}\cdot\vec{n}))\vec{n}$$

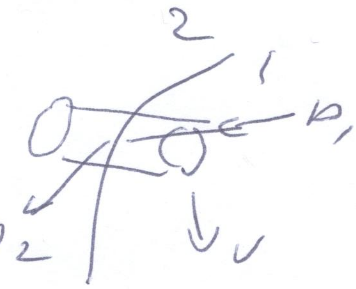
$$\Rightarrow \underbrace{(\vec{E} + 4\pi\vec{P})}_{\vec{D}} = 4\pi\sigma\,\vec{n}$$

for a linear medium $\vec{D} = \varepsilon\vec{E}$

$\vec{D}$ ← Displacement vector

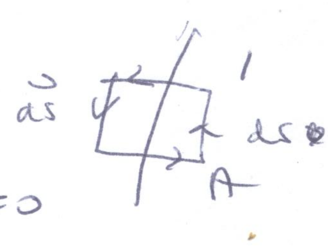
5b) Boundary conditions in a medium

$$\int_{\partial V} \vec{D}\cdot\vec{n}\,da = 4\pi\sigma A$$



$$\Rightarrow D_{2n} - D_{1n} = 4\pi\sigma$$

$$\vec{\nabla}\times\vec{E} = 0$$



$$\int_{\partial A} \vec{E}\cdot\vec{ds} = \int \vec{\nabla}\times\vec{E}\cdot\vec{dA} = 0$$

$$E^1_{tang} - E^2_{tang} = 0$$

Lecture #11

$$W_E = \frac{1}{2}\int d^3r\, d^3r' \, \frac{\rho(r)\,\rho(r')}{|\vec{r} - \vec{r}\,'|}$$

matter $P(r)$ is polarization
or dipole moment density

$$\phi = \int_{\partial V} \frac{\vec{P}(r')\cdot d\vec{a}}{|\vec{r} - \vec{r}\,'|} + \int dr' \, \frac{(-)\vec{\nabla}_{r'} P(r')}{|\vec{r} - \vec{r}\,'|}$$

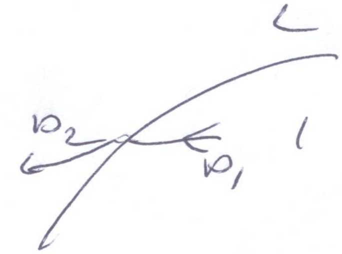
surface charge density $\sigma = \vec{P}\cdot\hat{n}$

induced charge density $\rho_b = -\vec{\nabla}_{r'}\vec{P}(r')$

$$\vec{\nabla}(E + 4\pi\vec{P}) = 4\pi\rho$$

also works for surface charge density

$$D_{2n} - D_{1n} = 4\pi\sigma$$



$$E_{1\,tang} = E_{2\,tang}$$

Today – Energy in medium
– dielectric tensor
– currents

Electrostatic energy in a medium

We do this again by building up a charge density

Work done

$$\delta W = \sum_i \delta q_i \, \phi(\vec{r}_i)$$

$$= \int d^3r \, \delta\rho(\vec{r}) \, \phi(\vec{r})$$

$$= \int d^3r \, \frac{\vec{\nabla}\delta\vec{D}}{4\pi} \, \phi(r)$$

$$= -\int d^3r \, \frac{\delta\vec{D}}{4\pi} \, \vec{\nabla}\phi$$

$$= \int d^3r \, \frac{\delta\vec{D}\,\vec{E}}{4\pi}$$

for a linear medium $\qquad \vec{D}_i = \varepsilon_{ij} E_j$

$$= \frac{1}{4\pi}\int d^3r \, \varepsilon_{ij} \, \delta E_j \, E_i$$

$$\varepsilon_{ij} = \varepsilon^S_{ij} + \varepsilon^A_{ij} \qquad \varepsilon^S_{ij} = \varepsilon_{ji}$$

$$\varepsilon^A_{ij} = -\varepsilon_{ji}$$

$$\frac{1}{4\pi}\int d^3r \, \varepsilon^S_{ij} \, \delta E_j \, E_i = \frac{1}{4\pi}\int d^3r \, \varepsilon^S_{ij} \, \frac{1}{2}\delta(E_i E_j)$$

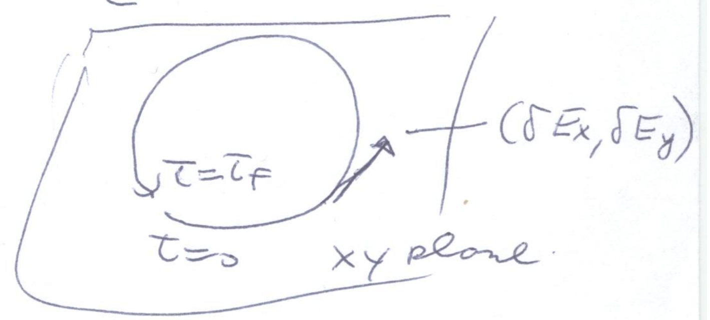
$$= \frac{1}{8\pi}\int d^3r \, \delta(\vec{D}^S \cdot \vec{E})$$

$$\Rightarrow W = \frac{1}{8\pi}\int d^3r \, \vec{D}^S \cdot \vec{E}$$

For the anti-symmetric part of the dielectric constant we obtain

$$\delta W = \frac{1}{4\pi}\int d^3r\, \varepsilon^A_{ij}\, \frac{1}{2}\left(\delta E_j \cdot E_i - \delta E_i \cdot E_j\right)$$

we now let the electric Field depend on a parameter τ



$$\vec{E} \to \vec{E}(\tau).$$

and calculate the work after $\vec{E}$ returns again to the same value.

$$\oint \delta W\, d\tau = \frac{1}{4\pi}\int d^3r\, \frac{\varepsilon^A_{ij}}{2} \oint d\tau \left(\delta E_j E_i - \delta E_i E_j\right)$$

we consider $\vec{E}$ in the xy plane.

$$= \frac{1}{4\pi}\int d^3r\, \frac{\varepsilon^A_{xy}}{2} \oint \left(\delta E_y E_x - \delta E_x E_y\right)$$

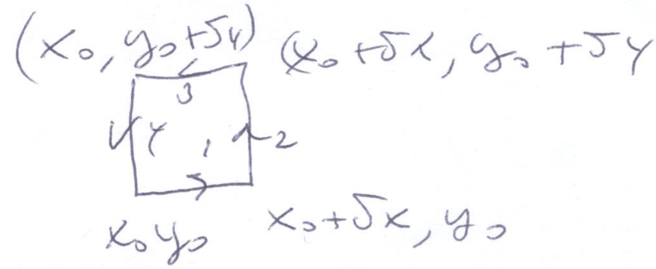
$$\oint \left(E_x\, dE_y - E_y\, dE_x\right)$$

This is in E_x, E_y space

Let's look how this work for

x, y ; $\frac{1}{2}\oint (x\, dy - y\, dx)$

We consider an infinitesimal loop V



$$\frac{1}{2}\int (x\,dy - y\,dx)$$

$$\frac{1}{2}\Big(\underset{①}{-y_0\,\delta x} + \underset{②}{(x_0+\delta x)\,\delta y}$$

$$(-)(-)\underset{③}{(y_0+\delta y)}\,\delta x \;\; \underset{④}{-\, x_0\,\delta y} \Big)$$

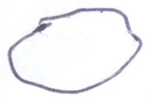
$$= \frac{1}{2}\; 2\,\delta x\,\delta y \;=\; \text{area of loop}$$

For bigger loop

integral over the inner loops cancels.

$\Rightarrow$ $\delta W \neq 0$ when $\vec{E}$ returns to its original value.

$\Rightarrow$ Σ_{ij}^{A} describes absorbtion



loop $\equiv$ $\begin{pmatrix} x(\tau) \\ y(\tau) \end{pmatrix} \equiv \vec{s}(\tau) \Rightarrow d\vec{s} = \begin{pmatrix} dx \\ dy \end{pmatrix}$

$$\oint x\,dy - y\,dx = \oint \begin{pmatrix} -y \\ x \end{pmatrix} \cdot \begin{pmatrix} dx \\ dy \end{pmatrix} =$$

$$= \int \left(\nabla \times \begin{pmatrix} -y \\ x \end{pmatrix}\right) da = 2A$$

a) Magnetostatics

Maxwell equations

$$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\partial_t\vec{B} \qquad \vec{\nabla}\cdot\vec{B} = 0$$

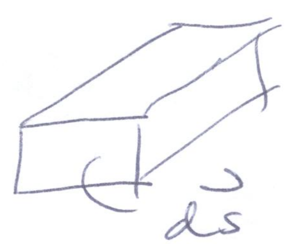
$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho \qquad \vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\partial_t\vec{E}$$

For statics we ignore the time derivatives. Then the eqs for $\vec{E}$ and $\vec{B}$ decouple:

Magnetostatics
$$\vec{\nabla}\cdot\vec{B} = 0$$
$$\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\underbrace{\vec{J}}_{\text{current density}}$$

b) Current Density

$$\vec{J} = \rho\vec{v}$$



$$\rho\vec{v}\cdot d\vec{S}$$

charge that flows through $d\vec{S}$ per second.

points outward

$$\oint_S d\vec{S}\cdot\vec{J} = -\frac{d}{dt}\int d^3r\,\rho$$

$$\oint_S d\vec{S}\cdot\vec{J} = \int d^3r\,\vec{\nabla}\cdot\vec{J}$$

$$\Rightarrow \vec{\nabla}\cdot\vec{J} + \frac{\partial\rho}{\partial t} = 0$$

continuity equation

Lecture #12

$$D_{2n} - D_{1n} = 4\pi\sigma$$

$$w = \frac{1}{8\pi}\int d^3r\, \vec{D}_s \cdot \vec{E}$$

$$\varepsilon_{ij} = \varepsilon_{ij}^{S} + \varepsilon_{ij}^{A}$$

$\uparrow$ energy loss

no energy loss $\Rightarrow$ ε is symmetric

magnetostatics : $\vec{\nabla}\cdot\vec{B} = 0$
$\vec{\nabla}\times\vec{B} = 0$

$$\vec{J} = \rho\vec{v} \qquad \vec{\nabla}\cdot J + \frac{\partial\rho}{\partial t} = 0$$

$$\int_S d\vec{S}\cdot\vec{J} = \int \operatorname{div}\vec{J}\, dV$$

$$= -\frac{d}{dt}\int \rho\, dV$$

Today – magnetic fields.
– Lorentz Force
– 3rd law

statics $\partial_t \rho = 0 \Rightarrow \vec{\nabla} \cdot \vec{j} = 0$

<u>Ohm's law</u> $\quad j_i = \sigma_{ij} E_j$

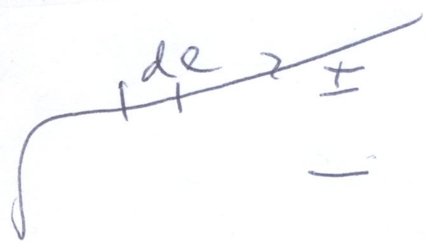
$\uparrow$ conductivity tensor

<u>Magnetic fields</u>;

1820 Oersted discovers that a current creates a magnetic field.

- no magnetic monopoles
- smallest unit is a dipole

c) <u>Experimental Results</u>



$$d\vec{F} = \frac{I}{c}(d\vec{l} \times \vec{B})$$

— force of current element

— field creates by a current

$$d\vec{B} = \frac{I}{c}\, d\vec{l} \times \frac{(\vec{x} - \vec{x}_e)}{|\vec{x} - \vec{x}_e|^3}$$

- Force on charge

$$I = \frac{dq}{dt} \Rightarrow I\, d\vec{l} = dq \frac{d\vec{l}}{dt} = dq\, \vec{v}$$

$$\Rightarrow d\vec{F} = \frac{dq}{c}\, \vec{v} \times \vec{B}$$

$\vec{F} = \frac{q}{c}\, \vec{v} \times \vec{B}$ Lorentz force

Field from a single charge with velocity $\vec{v}$

$$\vec{dB} = \frac{dq}{c}\, \vec{v} \times \frac{(\vec{r}-\vec{r}_q)}{|\vec{r}-\vec{r}_q|^3}$$

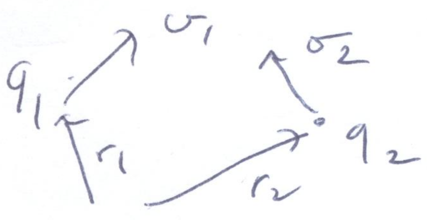
$$(dB)_i = \frac{dq}{c}\, \varepsilon_{ijk}\, v_j\, \frac{(\vec{r}-r_q)_k}{|\vec{r}-r_q|^3}$$

$$\partial_k \frac{1}{|\vec{r}-\vec{r}_q|}$$

$$\Rightarrow \partial_i dB_i = \frac{dq}{c}\, \underset{\substack{\uparrow \\ \text{anti-symmetric}}}{\varepsilon_{ijk}}\, v_j\, \underbrace{\partial_i \partial_k}_{\text{symmetric}} \frac{1}{|\vec{r}-\vec{r}_q|}$$

$$\Rightarrow \vec{\nabla} \vec{B} = 0.$$

d) Newton's 3rd law



$$\vec{F}_{21} = \frac{q_1}{c}\, \vec{v}_1 \times \vec{B}_2(r_1)$$
$$= \frac{q_1}{c}\, \vec{v}_1 \times \frac{q_2}{c}\left(\vec{v}_2 \times \frac{(\vec{r}_1-\vec{r}_2)}{|\vec{r}_1-\vec{r}_2|^3}\right)$$

$$\vec{F}_{12} = \frac{q_2}{c}\, v_2\, B_1(r_2)$$
$$= \frac{q_2}{c}\, \vec{v}_2 \times \frac{q_1}{c}\left(v_1 \times \frac{(\vec{r}_2-\vec{r}_1)}{(r_1-\vec{r}_2)^3}\right)$$

Newton's 3rd law is valid if $\vec{F}_{12} = -\vec{F}_{21}$ (34)

This is the case if

$$-\vec{v}_1 \times (\vec{v}_2 \times (\vec{r}_1 - \vec{r}_2)) = \vec{v}_2 \times (\vec{v}_1 \times (\vec{r}_2 - r_1))$$

$$\Rightarrow \quad \vec{v}_1 \times (v_2 \times \underbrace{(\vec{r}_1 - \vec{r}_2)}_{r})) = (\vec{v}_2 \times (\vec{v}_1 \times \underbrace{(\vec{r}_1 - r_2)}_{\vec{r}}))$$

outer product satisfies Jacobi identity

$$\vec{v}_1 \times (\vec{v}_2 \times \vec{r}) + \vec{v}_2 \times (\vec{r} \times \vec{v}_1) + \vec{r} \times (\vec{v}_1 \times \vec{v}_2) \Rightarrow$$

(it is not associative

$$- v_2 \times (\vec{v}_1 \times \vec{r})$$

$$\Rightarrow \quad \vec{r} \times (\vec{v}_1 \times \vec{v}_2) = 0.$$

Generally this is not true and Newton III is not valid. The reason is that the magnetic field contain momentum.

e) <u>$\vec{B}$ is a pseudovector</u>

$$\vec{B} = \frac{q}{c} \vec{v} \times \frac{\vec{r}}{r^2}$$

$\vec{v} \to -\vec{v}$ under space inversion.

$r \to -\vec{r}$ under space inversion

$\Rightarrow \vec{B} \to \vec{B}$ under space inversion

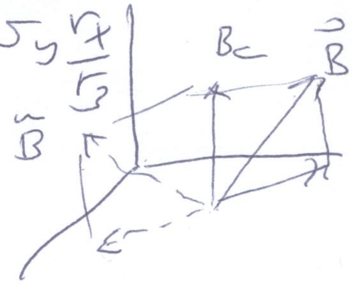
$\vec{B}$ is a pseudovector

Behavior of $\vec{B}$ under reflections

Say reflection in xy plan

then $z \to -z$ $\quad v_z \to -v_z$

$$B_z = \frac{q}{c} v_x \frac{r_y}{r^3} - \frac{q}{c} v_y \frac{r_x}{r^3}$$



$$\Rightarrow B_z \to B_z$$

$$B_x \to -B_x$$

$$B_y \to -B_y$$

Ampère's law

Next we derive Ampères law

$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{J}$$

$$\vec{J} \cdot d\vec{A} = I$$

$$\vec{J} \cdot d\vec{A}\, dl = I\, dl$$

choos $d\vec{A} \parallel \vec{J}$

$$\vec{J}\, d^3r = I\, d\vec{l}$$

$$\Rightarrow d\vec{B} = \vec{J} \frac{d^3r'}{c} \times \vec{\nabla}_r \frac{1}{|\vec{r} - r'|}$$

$$\Rightarrow \vec{B} = \int d^3r' \frac{\vec{J}}{c} \times \vec{\nabla}_r \frac{1}{|\vec{r} - \vec{r}'|}$$

$$\vec{\nabla} \times \vec{B} = \int d^3r' \left(\frac{\vec{\nabla} \cdot \vec{J}}{c} \vec{\nabla}_r \frac{1}{|\vec{r} - \vec{r}'|} - \frac{\vec{J}}{c} \vec{\nabla}_r \frac{1}{|\vec{r} - \vec{r}'|} \right.$$

<u>statics</u> $\vec{\nabla}\vec{j} + \underset{\overset{\shortparallel}{0}}{\frac{\partial \rho}{\partial t}} = 0 \Rightarrow \vec{\nabla}\vec{j} = 0$

$$\vec{\nabla}_r \frac{1}{|\vec{r} - r'|} = -4\pi\, \delta^3(r - r')$$

$$\Rightarrow \vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{j}$$

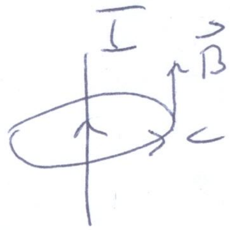
integral form of Ampère's law:

$$\oint_S \vec{\nabla} \times \vec{B} \cdot d\vec{s} = \frac{4\pi}{c} \int_S \vec{j} \cdot d\vec{s}$$

$$\shortparallel$$

$$\oint_c \vec{B} \cdot d\vec{\ell}$$

$$\Rightarrow \oint_c \vec{B} \cdot d\ell = \frac{4\pi}{c} \sum_{\text{enclosed}} I .$$



$$B\, 2\pi r = \frac{4\pi}{c} I$$

Lecture # 13

$$d\vec{B} = \frac{I}{c}\, d\vec{l} \times \frac{\vec{x} - \vec{x}_l}{|\vec{x} - \vec{x}_l|^3}$$

$$d\vec{F} = \frac{I}{c}\,(d\vec{l} \times \vec{B}$$

$$d\vec{F} = \frac{da}{c}\; \vec{v} \times \vec{B})$$

- $\vec{\nabla} \vec{B} = 0$
- Newton III is not valid
- $\vec{B}$ is pseudovector
- Ampère $\quad \vec{\nabla} \times \vec{B} = \frac{4\pi}{c}\vec{j}$

$$\oint_c \vec{B} \cdot d\vec{l} = \frac{4\pi}{c} \sum I_{\text{enclosed}}$$

Today
- Gauge potential
- multipole expansion
- magnetic moment

Gauge Potential

$$\vec{B}_i = \frac{1}{c}\int d^3r'\, \vec{j}(\vec{r}\,')\times(-\vec{\nabla}_r)\frac{1}{|\vec{r}-\vec{r}\,'|}$$

$$= \frac{1}{c}\int d^3r'\, \varepsilon_{ijk}\, j_j\, \partial_k \frac{-1}{|r-r'|}$$

$$= \frac{1}{c}\varepsilon_{ijk}\partial_k \int d^3r'\, j_j \frac{-1}{|r-r'|}$$

$$= +\frac{1}{c}\varepsilon_{ikj}\partial_k \int d^3r'\, j_j \frac{1}{|\vec{r}-r'|}$$

$$= + \quad \vec{\nabla}\times\underbrace{\frac{1}{c}\int d^3r'\, j_j \frac{1}{|r-r'|}}_{\vec{A}}$$

$\vec{A}$ gauge potential

$$\vec{A} = \frac{1}{c}\int d^3r'\, \vec{j}(\vec{r}\,')\frac{1}{|\vec{r}-\vec{r}\,'|}$$

$\vec{A}$ is not unique $\quad \vec{A}\to\vec{A}+\vec{\nabla}\psi$

gives the same $\vec{B}$ field because $\vec{\nabla}\times\vec{\nabla}\psi = 0$

This is called gauge invariance

<u>gauge theories</u>: em force
strong force
weak force

<u>Example</u>: vector potential of constant $\vec{B}$ field $\quad \vec{B} = (0,0,B_z)$

$A_1 = B_0\,(0, x, 0)$

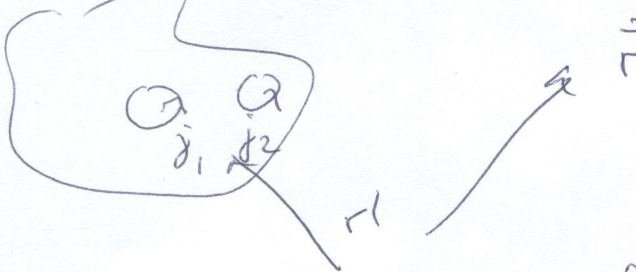
$A_2 = B_0(-y, 0, 0)$

$A_3 = \frac{B_0}{2}(-y\, x, 0)$

$$A_1 - A_3 = B_0\left(\frac{y}{2}, \frac{x}{2}, 0\right)$$
$$= \frac{B_0}{2}\vec{\nabla}(XY)$$

Multpole expansion

Since $A(r) = \frac{1}{c}\int d^3r' \frac{\vec{J}(r')}{|r-r'|}$



we can again do the multipole expansion

$$\frac{1}{|\vec{r}-r'|} = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \frac{r'^{\ell}}{r^{\ell+1}}\frac{4\pi}{2\ell+1} Y_{\ell m}(\theta,\varphi)\, Y^{*}_{\ell m}(\theta',\varphi')$$

$$\Rightarrow \bar{A}(r) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \frac{4\pi}{r^{\ell+1}(2\ell+1)} Y_{\ell m}(\theta,\varphi)\int d^3r'\, r'^{\ell}\, Y^{*}_{\ell m}(\theta',\varphi') \times \vec{j}(r')$$

regular product

$\ell=0$ then $\int d^3r'\, \vec{j}(r')$

consider $j_1 = -\int d^3x\, x_1\, \partial_1 j_1$

$\vec\nabla \vec{j} = 0 \qquad = -\int d^3x\, x_1(-\partial_2 j_2 - \partial_3 j_3)$

total deriativ

$= 0$

$\Rightarrow$ vector potential of current $\propto \frac{1}{r^2}$

$$\sum_{m=-\ell}^{\ell} Y_{\ell m}(\theta,\varphi)\, Y^{*}_{\ell m}(\theta',\varphi') \overset{\ell=1}{=}$$

$$\frac{3}{8\pi}(+)\frac{(x+iy)}{r}\frac{(x'-iy')}{r'} + \frac{3}{4\pi}\frac{z z'}{rr'} + \frac{3}{8\pi}\frac{(x-iy)}{r}\frac{(x'+iy')}{r'}$$

$$= -\frac{3}{8\pi}\frac{1}{rr'}\left(xx' + yy' - \cancel{ixy'} + \cancel{iyx'} + 2zz' + xx' + yy' + \cancel{ixy'} - \cancel{iyx'}\right)$$

$$= -\frac{3}{4\pi}\frac{1}{rr'}(xx' + yy' + zz') = +\frac{3}{4\pi}\frac{\vec{r}\cdot\vec{r}'}{|\vec{r}|\,|\vec{r}'|}$$

So the gauge potential of the $\ell=1$ multipol is given by

$$\vec{A}^{\ell=1}(r) = \frac{4\pi}{3}\frac{1}{r^2}\left(+\frac{3}{4\pi}\right)\int d^3r'\, r' \frac{(\vec{r}\cdot\vec{r}')}{r r'}\, \vec{j}(r')$$

$$= +\frac{1}{r^3}\int d^3r' (\vec{r}\cdot\vec{r}')\, \vec{j}(r')$$

$-\frac{\vec{r}}{r^3} = \vec{\nabla}_r \frac{1}{r}$ $\qquad = -\int d^3r'\, \vec{\nabla}\frac{1}{r}\cdot\vec{r}'\, \vec{j}(r')$

$\nabla\vec{j} = 0$

we want to rewrite this in term of a magnetic moment density $\vec{M} = \frac{\vec{r}\times\vec{j}}{2c}$

magnetic moment $\vec{m} = \int d^3r' \frac{\vec{r}'\times j(r')}{2c}$

Let us calculate $\vec{\nabla}\frac{1}{r}\times\vec{m} =$

$$= \frac{1}{2c}\vec{\nabla}\frac{1}{r}\times\int d^3r'(r'\times j)$$

$$= \frac{1}{2c}\int d^3r' \underbrace{\vec{\nabla}\frac{1}{r}\times(r'\times j)}$$

$$-\underset{(A)}{\left(\vec{\nabla}\frac{1}{r}\cdot r'\right)}\vec{j} + \underset{(B)}{\left(\vec{\nabla}\frac{1}{r}\cdot j\right)}\vec{r}'$$

$\delta_{ji}+\delta_{jk}$

$$\int j_j\, \partial_j (r_i' r_k') = -\int \underbrace{\partial_j' j_j}_{=0} r_i' r_k' = 0$$

$\int j_k r_i' + \int j_i r_k' = 0$ $\Rightarrow$ term (A) and (B) give the same result

(B) $\Rightarrow -\frac{1}{2c}\int d^3r' + \frac{r_k}{r^3}\cdot j_k r_i' = +\frac{1}{2c}\int d^3r \frac{r_k}{r^3} j_i r_k'$

$\Rightarrow$ (A) term gives the same result.

$$\Rightarrow \vec{A}^{\ell=1} = \vec{\nabla}\frac{1}{r}\times\vec{m}$$

Lecture # 14

$$\vec{B} = \frac{1}{c}\int d^3r' \, j(r') \left(-\vec{\nabla}_r\right) \frac{1}{|\vec{r}-\vec{r}'|}$$

$$A = \frac{1}{c}\int d^3r' \, \vec{j}(r') \frac{1}{|\vec{r}-\vec{r}'|}$$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

invariance $A \to A + \vec{\nabla}\varphi$

multipole expansion

$$\vec{A}(r) = \sum_{lm} \frac{4\pi}{r^{l+1}} \frac{1}{(2l+1)} Y_{lm}(\theta,\varphi) \int d^3r' \, r'^l \, Y^*_{lm}(\theta',\varphi')$$

$l=0$ term vanishes

$$\sum_{m=-l}^{l} Y_{lm}(\theta,\varphi) \, Y_{lm}(\theta',\varphi') = \frac{3}{4\pi} \frac{\vec{r}\cdot\vec{r}'}{r r'}$$

$l=1$

$$A_{l=1} = -\int d^3r' \, \vec{\nabla}\frac{1}{r} \cdot \vec{r}' \, j(r')$$

magnetic moment density $\vec{M} = \frac{\vec{r} \times \vec{j}}{2j}$

Today – $A^{l=1}$

– magnetic field in matter

√

The $\ell=1$ magnetic field is given by

$$B_i^{\ell=1} = \vec{\nabla} \times A^{\ell=1} = \vec{\nabla} \times \left(\vec{\nabla}\frac{1}{r} \times \vec{m}\right)$$

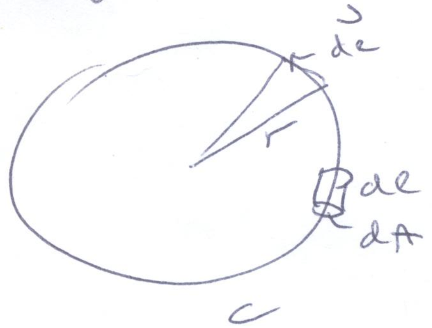
$$= \underbrace{\left(\vec{\nabla}^2 \frac{1}{r}\right)}_{=0} \vec{m} + \vec{\nabla}_k\left(\left(\vec{\nabla}_i \frac{1}{r}\right) m_k\right)$$

m_k is an integral and does not depend on $\vec{r}$

$$\Rightarrow B_i = +\partial_i\left(\partial_k \frac{1}{r} m_k\right) = -\partial_i \underbrace{\frac{\vec{m}\cdot\vec{r}}{r^3}}_{\text{magnetic potential}}$$

<u>Examples</u>

a) magnetic moment of current loop



$$\vec{m} = \frac{1}{2c}\int d^3r\, \vec{r} \times \vec{j}$$

$$\vec{j}\, d^3r = \vec{j}\cdot d\vec{A}\, \vec{d\ell}$$

$$\vec{m} = \frac{I}{2c}\oint \vec{r} \times \vec{d\ell} = \frac{I A}{2c} \leftarrow \text{area}$$

b) magnetic moment of orbiting point charge

$$\vec{j} = \delta^3(\vec{r} - \vec{r}_p)\, q\, \vec{v}_p$$

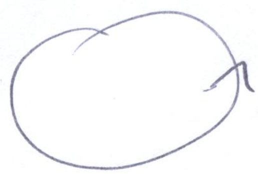
$$\vec{m} = \frac{1}{2c}\int d^3r\, \vec{r} \times \delta(\vec{r} - \vec{r}_p)\, q\, \vec{v}_p = \frac{q}{2c}\vec{r}_p \times \vec{v}_p$$

$$= \frac{q}{2mc} L_p \leftarrow \text{angular momentum}$$

Lecture #15

$$A^{\ell=1} = \vec{\nabla}\frac{1}{r} \times \vec{m}$$

$$\vec{B} = -\vec{\nabla}\,\frac{\vec{m}\cdot\vec{r}}{r^3}$$



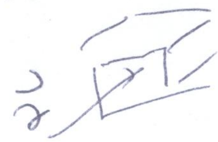
I $\quad \vec{m} = \frac{I A}{c}$

correct

~~corr~~

$$\vec{m}_p = \frac{q}{mc} L_p$$

Today Magnetic Field in matter

$\vec{m} = \frac{1}{2c}\int d^3r'\, \vec{r}' \times \vec{j}$  $\rho\, d\varphi$

$$\vec{j}\cdot d^3r = I\,\delta(\rho-\rho_1)\,\delta(z)\,d\vec{\ell}\,d\rho\,dz$$

$\frac{qv}{\ell^3} = \frac{I}{\ell^2}$ $\quad \frac{1}{2c}\int d^3r\, \vec{r}' \times \vec{j}$

$$= \int \frac{I}{2c}\rho_0\, 2\pi\rho_0 = \frac{I\pi\rho_0^2}{c} = \frac{IA}{c}$$

Magnetic fields in matter

model: the effect of an external magnetic field can be derived by a magnetic dipole density $\vec{M}$

Field of a single dipole $\vec{A}_i = -m_i \times \nabla_r \frac{1}{|\vec{r}-\vec{r}_i|}$

$$\Rightarrow A_{tot} = \sum_i -m_i \times \vec{\nabla}_r \frac{1}{|\vec{r}-\vec{r}_i|}$$

$\phi = \frac{1}{c}\int d^3r' \frac{J(r')}{|r-r'|}$

$$= -\int d^3r' \, \vec{M}(r') \times \vec{\nabla}_r \frac{1}{|r-r'|}$$

$$\vec{\nabla}_r = -\vec{\nabla}_{r'}$$

$$= -\int d^3r' \left(\vec{\nabla}_{r'} \times \left(\vec{M}(r') \frac{1}{|\vec{r}-\vec{r}'|} \right) - \frac{1}{|r-r'|} \vec{\nabla}_{r'} \times M(r') \right)$$

$$= \int_{\partial V} \frac{1}{|\vec{r}-\vec{r}'|} \vec{M} \times \vec{n} \, da + \int_V d^3r' \frac{1}{|r-r'|} \vec{\nabla}_{r'} \times M(r')$$

$\int_V \vec{\nabla} \times \vec{A} \, d^3r' = \int_{\partial V} \vec{n} \times \vec{A} \, da$

induced current density $c \vec{\nabla} \times \vec{M}$

surface current $c \vec{M} \times \vec{n} \, \delta(f(r)-)$

$$\Rightarrow \vec{\nabla} \times \vec{B} = \frac{4\pi}{c}(\vec{J} + J_{ind})$$

$$= \frac{4\pi}{c}\left(\vec{J} + c(\vec{\nabla} \times M) + c \, \delta(f(r)-) \, \nabla \vec{M} \times \vec{n}\right)$$

surface current

$\vec{r}_s$ defines the surface

S

$$\Rightarrow \left(\vec{\nabla} \times B - 4\pi\, \vec{\nabla} \times \vec{M}\right) = \frac{4\pi}{c}\vec{j}$$

For induced volume current

magnetic Field $\vec{H} = \vec{B} - 4\pi \vec{M}$

$\vec{j}$ is the current due to the free charges

$\vec{B}$ is called the magnetic induction

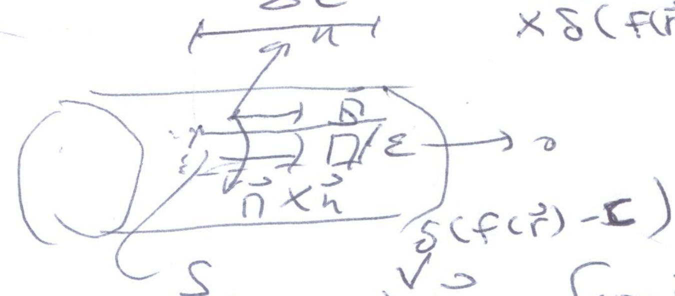
For the effect of the surface current we start from the integral form of Ampère's law.

$$\int \vec{\nabla} \times \vec{B} \cdot d\vec{S} = \frac{4\pi}{c}\int \vec{j} \cdot d\vec{S} + \frac{4\pi}{c}\int c(\vec{M} \times \vec{n})\, d\vec{S}$$

$\cancel{\delta(\vec{r} - \vec{r}_s)}$ $\times\, \delta(f(\vec{r}) - c)$

$\parallel$ stokes

$$\int \vec{B} \cdot d\vec{l}$$



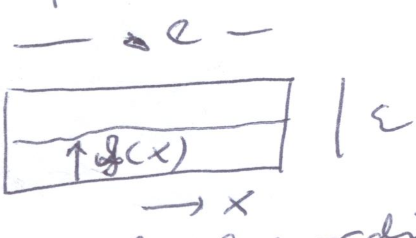
$\vec{B} = 0$ on outside

$$\frac{4\pi}{c}\int c(\vec{M} \times \vec{n})\, d\vec{S} = \int 4\pi \vec{M}\, d\vec{l}$$

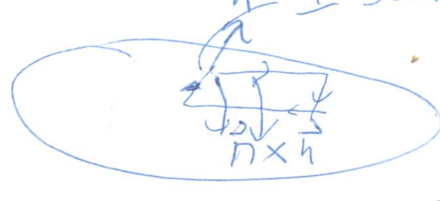
$$\Rightarrow \int (\vec{B} - 4\pi\vec{M})\, d\vec{l} = \frac{4\pi}{c}\int \vec{j} \cdot d\vec{S}$$

$$= (\vec{B} - 4\pi\vec{M}) \cdot \Delta\vec{l}$$

This is important for permanent magnets



$\vec{n} \perp$ surface



$d\vec{S} \parallel \vec{M} \times \vec{n}$

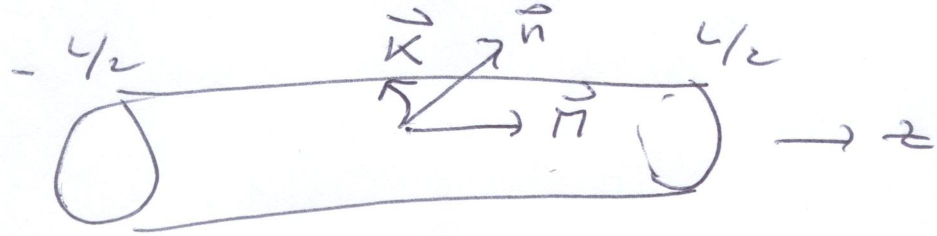
x, y local coordinates

$$\frac{4\pi}{c}\int c\, \vec{M} \times \vec{n}\, \delta(f(\vec{r}) - c)\, d\vec{S} = 4\pi \int_{\Delta l} dl\, \vec{M} \times \vec{n} \underbrace{\int dy\, \delta(y - f(x))}_{1}$$

$$= 4\pi \vec{M} \cdot \Delta\vec{l}$$

Example

permanent magnet in cylinder shape



$$\vec{M}(r) = M\hat{z} = \text{constant}$$

$$\Rightarrow \quad c\,\vec{\nabla}\times\vec{M} = 0 \text{ in bulk.}$$

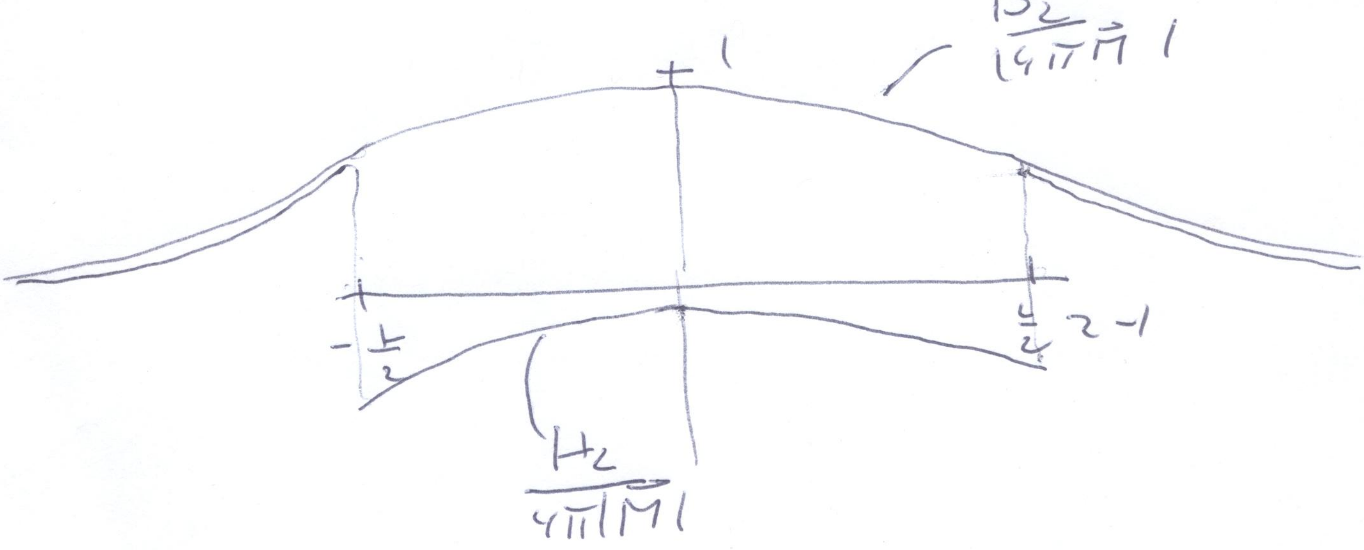
surface current $\quad K = c\vec{M}\times\vec{n}$

$$\vec{j} = 0 \Rightarrow \vec{\nabla}\times\vec{H} = 0$$

$$\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\big(\underbrace{c\,\vec{\nabla}\times\vec{M}}_{=0} + c\vec{M}\times\vec{n}\big)$$

$$|z| < \frac{L}{2} \qquad \frac{H_z}{4\pi|\vec{M}|} = \frac{|B_z|}{4\pi|\vec{M}|} - 1$$

$$|z| > \frac{L}{2} \qquad \frac{H_z}{4\pi\vec{M}} = \frac{B_z}{4\pi M}$$



Magnets

a) Permanent magnets

material with fixed magnetic dipole density $\vec{M}(\vec{r})$

$$\vec{J}_{free} = 0 \quad \Rightarrow \quad \vec{\nabla} \times \vec{H} = 0$$

always. $\vec{\nabla} \cdot \vec{B} = 0$

$$\Rightarrow \vec{\nabla} \cdot \vec{H} = 4\pi \left(-\vec{\nabla} \cdot \vec{M} \right)$$

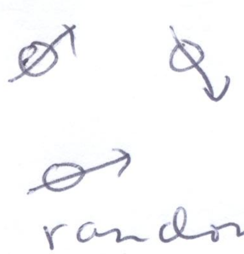
$\Rightarrow -\vec{\nabla} \cdot \vec{M}$ is the effective monopole density

$$\vec{\nabla} \times \vec{H} = 0 \quad \Rightarrow \quad \exists\, \phi_M \mid \vec{H} = -\vec{\nabla} \phi_M$$

$$\vec{\nabla}^2 \phi_M = 4\pi\, \vec{\nabla} \cdot \vec{M}$$

However the surface current discussed in the previous section is the most important

b) Paramagnets



atoms with a permanent magnetic dipole moment $\vec{\mu}$

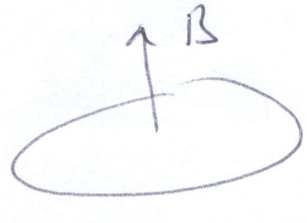
random for $\vec{B}_0 = 0$

oriented for $\vec{B}_0 \neq 0$

$$\vec{B} = \mu \vec{H}$$

$\mu > 1$ because the magnetic moments contribute to $\vec{B}$

c) Diamagnetism $\vec{\mu} = 0$



$\vec{\mu}_d = \frac{e\vec{L}}{2mc}$ when we have $\vec{L} = 0$

Lenz : electrons try to oppose an increasing $\vec{B}$

$\Rightarrow$ μ induced is opposite to $\vec{B}$

Generally this is a very small effect