Lecture #4

Gauss $\vec{\nabla} E = 4\pi\rho$

Poisson $\vec{\nabla}^2 \phi = -4\pi\rho$

Laplace $\vec{\nabla}^2 \phi = 0$

Today: Uniqueness

Solution of Laplace equation

Integral form of Gauss law

$$\int_V dv\, \vec{\nabla}\cdot\vec{E} = \int dV\, 4\pi\rho$$

$$\|$$

$$\int_{\partial V} da\, \vec{n}\cdot\vec{E}$$

e) <u>Poisson law</u>

$$\left.\begin{aligned} \vec{\nabla}\vec{E} &= 4\pi\rho \\ \vec{E} &= -\vec{\nabla}\phi \end{aligned}\right\} \Rightarrow \vec{\nabla}^2\phi = -4\pi\rho$$

Poisson equatia

$\rho=0$ $\quad \vec{\nabla}^2\phi = 0$ $\quad$ Laplace eq

solutions are called harmonic function.

f) <u>Uniqueness</u>

A vector field that satisfies

$$\vec{\nabla}\vec{E} = 0 \qquad \vec{\nabla}\times\vec{E}$$

and vanishes at least as $\frac{1}{r^2}$ for $r\to\infty$

vanishes everywhere

This follows from liouville's theoreme:

if f is a harmonic function on $\mathbb{R}^n$

and f is bounded from above or below

then f is constant

Reason $\vec{E} = -\vec{\nabla}\phi$

$$\vec{\nabla}^2\phi = 0$$

$\vec{E} \sim \frac{1}{r^2}$ for $r \to \infty \Rightarrow \phi \sim \frac{1}{r}$ for $r \to \infty$

$\vec{\nabla}\cdot\vec{E} = 0 \Rightarrow \exists$ no singularities

$\Rightarrow \phi$ is bounded $\Rightarrow \phi =$ constant

$\Rightarrow \vec{E} = 0$

Consequence: uniqueness theorem

$$\left.\begin{aligned}\vec{\nabla}\cdot\vec{E}_1 &= 4\pi\rho_1\\ \vec{\nabla}\cdot\vec{E}_2 &= 4\pi\rho_2\end{aligned}\right\} \Rightarrow \vec{\nabla}(\vec{E}_1 - E_2) = 0$$

statics $\Rightarrow \vec{\nabla}\times(\vec{E}_1 - \vec{E}_2) = 0$

$\Rightarrow E_1 - E_2 = 0$

Solutions of the Laplace eq

$\vec{\nabla}^2\phi = 0$

$z = r\cos\theta$
$x = r\sin\theta\cos\varphi$
$y = r\sin\theta\sin\varphi$

spherical coordinates

$$\vec{\nabla}^2\phi = \frac{1}{r}\partial_r^2(r\phi) + \frac{1}{r^2\sin\theta}\partial_\theta\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2}$$

$$\phi = \frac{u(r)}{r}P(\theta)\,Q(\varphi)$$

sepration of variables

$$\nabla^2\phi = \frac{P\,Q}{r}\,\partial_r^2 u + \frac{u\,Q}{r r^2 \sin\theta}\,\partial_\theta(\sin\theta\,\partial_\theta P)\,g$$

$$+ \frac{u\,P}{r r^2 \sin^2\theta}\,\frac{\partial^2 Q}{\partial \varphi^2} = 0$$

divide by $u P Q$

$$\frac{u^{-1}}{r}\partial_r^2 u + \frac{P^{-1}}{r r^2 \sin\theta}\,\partial_\theta(\sin\theta\,\partial_\theta P)$$

$$+ \frac{Q^{-1}}{r r^2 \sin^2\theta}\,\frac{\partial^2 Q}{\partial \varphi^2} = 0$$

$\times r^3$

$$r^2 u^{-1}\partial_r^2 u + \frac{P^{-1}}{\sin\theta}\,\partial_\theta(\sin\theta\,\partial_\theta P)$$

$$+ \frac{Q^{-1}}{\sin^2\theta}\,\frac{\partial^2 Q}{\partial \varphi^2} = 0$$

only depends on r

only depends on θ, φ

$\Rightarrow$ both should be constant because the equation is valid for all r, θ, φ

$$r^2 u^{-1}\partial_r^2 u = c \Rightarrow \partial_r^2 u = c\frac{u}{r^2}$$

$$u = \alpha r^{\ell+1} \Rightarrow \ell(\ell+1)\, r^{\ell-2} = c\, r^{\ell-2}$$

$$\Rightarrow c = \ell(\ell+1)$$

$\frac{r_0}{V}$

then

$\ell(\ell+1) \sin^2\theta + P^{-1} \sin\theta\, \partial_\theta(\sin\theta\,\partial_\theta P)$

$+ Q^{-1} \frac{\partial Q^2}{\partial \varphi^2} = 0$

$\Rightarrow \ell(\ell+1)\sin^2\theta + P^{-1}\sin\theta\,\partial_\theta(\sin\theta\,\partial_\theta P)$

$= -\left(\frac{\partial Q^2}{\partial \varphi^2}\right) Q^{-1}$

function of θ function of φ

equal $\forall\, \theta, \varphi \Rightarrow$ they have to be constant

$\Rightarrow Q^{-1} \frac{\partial^2 Q^2}{\partial \varphi^2} = -m^2 \Rightarrow Q = e^{\pm i m \varphi}$

$\Rightarrow \sin\theta\,\partial_\theta(\sin\theta\,\partial_\theta P) = \ell(\ell+1)\sin^2\theta\, P - m^2 P$

This equation is known as the Legendre equation. $P = P(\cos\theta)$

The solutions are given by the associated Legendre polynomials

The combination PQ is given by $PQ = c' P_{\ell m}(\cos\theta)\, e^{im\varphi} \equiv c' Y_{\ell m}(\theta,\varphi)$

spherical harmonics

The constant c' is fixed by the normalization sum rule

$$\sum_{m=-\ell}^{\ell} Y_{\ell m}(\theta,\varphi)\, Y^{*}_{\ell m}(\theta,\varphi) = \frac{2\ell+1}{4\pi}$$

$$Y_{00} = \frac{1}{\sqrt{4\pi}}$$

$$Y_{11} = -\sqrt{\frac{3}{8\pi}} \sin\theta\, e^{i\phi} = -\sqrt{\frac{3}{8\pi}}\,\frac{x+iy}{r}$$

$$Y_{10} = \sqrt{\frac{3}{4\pi}} \cos\theta = \sqrt{\frac{3}{4\pi}}\,\frac{z}{r}$$

$$Y_{1-1} = \sqrt{\frac{3}{8\pi}} \sin\theta\, e^{-i\varphi} = \sqrt{\frac{3}{8\pi}}\,\frac{x-iy}{r}$$

We have found an infinite set of harmonic functions

$$\nabla^2 r^{\ell}\, Y_{\ell m}(\theta,\varphi) = 0$$

$$\ell(\ell+1) = (-\ell-1)((-\ell-1)+1)$$

$\Rightarrow u(r) = \frac{1}{r^{\ell}}$ is also solution

$$\Rightarrow \quad \vec{\nabla}^2 \frac{1}{r^{\ell+1}} Y_{\ell m}(\theta, \varphi) = 0$$

Most general solution of the Laplace eq.

$$\phi(r,\theta,\varphi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left(A_{\ell m} r^{\ell} + B_{\ell m} r^{-\ell-1} \right) Y_{\ell m}(\theta, \varphi)$$

This is a complete set of functions

Strategy for solution of Poisson eq

$$\nabla^2 \phi = -4\pi\rho$$

then $\nabla^2(\phi + \phi_h) = -4\pi\rho$

special solution

$$\vec{\nabla}^2 \phi_h = 0$$

Generally we have a charge distribution and boundary conditions

The ϕ_h are determined by the boundary conditions.

outside charge distribution; $\rho = 0$

$\Rightarrow \vec{\nabla}^2 \phi = 0$ $\qquad \phi \to 0$ for $r \to \infty \Rightarrow A_{\ell m} = 0$

inside charge distribution: solution should be regular $\Rightarrow B_{\ell m} = 0$

Lecture # 5

$$\vec{\nabla}^2 \phi = 0$$

$$\vec{\nabla}^2 \phi = \frac{1}{r}\partial_r^2 (r\phi) + \frac{1}{r^2 \sin\theta}\partial_\theta \sin\theta \frac{\partial \phi}{\partial \theta} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \phi}{\partial \varphi^2}$$

$$\phi = \frac{u(r)}{r} P(\theta)\, Q(\varphi)$$

$$u(r) = \begin{cases} r^{\ell+1} \\ \dfrac{1}{r^\ell} \end{cases} \qquad Q(\varphi) = e^{im\varphi}$$

$$+\sin\theta\, \partial_\theta (\sin\theta\, \partial_\theta P) + \ell(\ell+1)\sin^2\theta\, P - m^2 P = 0$$

$P = P(\cos\theta)$ is equation for Legendre polynomials

$$P = \cos\theta \Rightarrow \sin\theta\, \partial_\theta (\sin\theta\, \partial_\theta P) = -2\sin^2\theta \cos\theta$$

$\Rightarrow$ solution for $\ell = 1$ and $m = 0$.

Today: - Uniqueness
- Examples.

Some properties of spherical harmonics

$$Y_{\ell m}(\theta,\varphi) = \sqrt{\frac{2\ell+1}{4\pi}}\sqrt{\frac{(\ell-m)!}{(\ell+m)!}}\, P_{\ell m}(\cos\theta)\, e^{im\varphi}$$

$$r^2 \vec{\nabla}^2 Y_{\ell m}(\theta,\varphi) = -\ell(\ell+1) Y_{\ell m}(\theta,\varphi)$$

$$Y_{\ell -m}(\theta,\varphi) = (-1)^m Y^*_{\ell m}(\theta,\varphi)$$

orthogonality

$$\int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta\, d\theta\; Y^*_{\ell' m'}(\theta,\varphi)\, Y_{\ell m}(\theta,\varphi) = \delta_{\ell\ell'}\delta_{mm'}$$

completeness

$$\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Y^*_{\ell m}(\theta',\varphi')\, Y_{\ell m}(\theta,\varphi) = \delta(\varphi-\varphi') \times \delta(\cos\theta - \cos\theta')$$

$\Rightarrow$ An arbitrary function of θ and φ can be expressed as

$$f(\theta,\varphi) = \sum a_{\ell m} Y_{\ell m}(\theta,\varphi)$$

sum rule

$$\sum_{m=-\ell} Y_{\ell m}(\theta,\varphi)\, Y^*_{\ell m}(\theta,\varphi) = \frac{2\ell+1}{4\pi}$$

$r < a \qquad \phi = \sum A_{\ell 0}\, r^\ell\, Y_{\ell 0}(\theta, \varphi)$

$r > a \qquad \phi = \sum B_{\ell 0} \frac{1}{r^{\ell+1}} Y_{\ell 0}(\theta, \varphi)$

$$Y_{\ell 0}(0, \varphi) = \sqrt{\frac{2\ell+1}{4\pi}}$$

$r < a$ on z axis $\qquad \phi(z) = \frac{q}{a-z} = \sum_{n=0}^{\infty} \frac{q}{a}\left(\frac{z}{a}\right)^n$

$\Rightarrow \quad A_{\ell 0} = \frac{q}{a^{\ell+1}} \sqrt{\frac{4\pi}{2\ell+1}}$

$r > a$ on z axis $\qquad \phi_z = \frac{q}{z-a} = q \sum_{n=0}^{\infty} \frac{a^n}{z^{n+1}}$

$\Rightarrow \quad B_{\ell 0} = q a^\ell \sqrt{\frac{4\pi}{2\ell+1}}$

potential in entire space is then given by

$$\phi(r < a) = \sum_{\ell=0}^{\infty} \frac{q r^\ell}{a^{\ell+1}} \sqrt{\frac{4\pi}{2\ell+1}} Y_{\ell 0}(\theta, \varphi)$$

$$\phi(r > a) = \sum_{\ell=0}^{\infty} \frac{q a^\ell}{r^{\ell+1}} \sqrt{\frac{4\pi}{2\ell+1}} Y_{\ell 0}(\theta, \varphi)$$

do not depend on φ

Example 1

charge q at $r=0$

$$\rho(r) = q\,\delta^3(r)$$

potential is spherically symmetric

$\Rightarrow$ only $l=0$ and $m=0$ are allowed

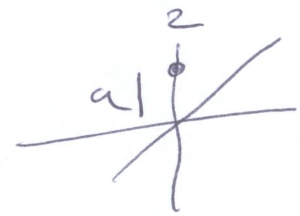
$$\Rightarrow \phi = A_{00} + \frac{B_{00}}{r}$$

$\phi \to 0$ for $r \to \infty$ $\Rightarrow$ $A_{00} = 0$

Example 2

charge q at $z=a$

$$\phi = \sum \left(A_{lm} r^l + \frac{B_{lm}}{r^{l+1}} \right) Y_{lm}(\theta,\varphi)$$



axial symmetry $\Rightarrow$ $m=0$

We split the space in $r<a$ and $r>a$

$r<a$; potential is finite $\Rightarrow$ $B_{lm} = 0$

$r>a$; potential vanishes for $r \to \infty$

$\Rightarrow$ $A_{lm} = 0$

We are going to determine the coëfficients from the potential on the z axis where it is given by $\phi(z) = \dfrac{q}{|z-a|}$

$r < a \qquad \phi = \sum A_{\ell 0}\, r^\ell\, Y_{\ell 0}(\theta, \varphi)$

$r > a \qquad \phi = \sum B_{\ell 0} \frac{1}{r^{\ell+1}} Y_{\ell 0}(\theta, \varphi)$

$$Y_{\ell 0}(0, \varphi) = \sqrt{\frac{2\ell+1}{4\pi}}$$

$r < a$ on z axis $\qquad \phi(z) = \frac{q}{a-z} = \sum_{n=0}^{\infty} \frac{q}{a}\left(\frac{z}{a}\right)^n$

$$\Rightarrow \quad A_{\ell 0} = \frac{q}{a^{\ell+1}} \sqrt{\frac{4\pi}{2\ell+1}}$$

$r > a$ on z axis $\qquad \phi_z = \frac{q}{z-a} = q \sum_{n=0}^{\infty} \frac{a^n}{z^{n+1}}$

$$\Rightarrow \quad B_{\ell 0} = q a^\ell \sqrt{\frac{4\pi}{2\ell+1}}$$

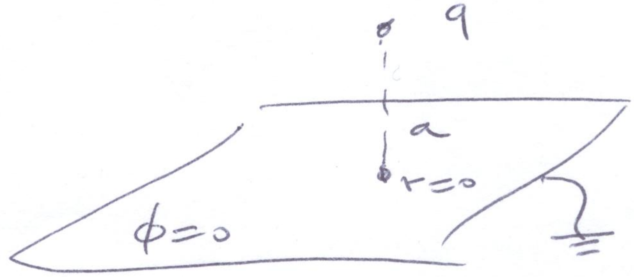
potential in entire space is then given by

$$\phi(r < a) = \sum_{\ell=0}^{\infty} \frac{q r^\ell}{a^{\ell+1}} \sqrt{\frac{4\pi}{2\ell+1}}\, Y_{\ell 0}(\theta, \varphi)$$

$$\phi(r > a) = \sum_{\ell=0}^{\infty} \frac{q a^\ell}{r^{\ell+1}} \sqrt{\frac{4\pi}{2\ell+1}}\, Y_{\ell 0}(\theta, \varphi)$$

do not depend on φ

Example 3



ϕ is axially symmetric $\Rightarrow m = 0$

again split space in $r < a$ and $r > a$

$r < a$ potential is finite $\Rightarrow B_{em} = 0$

$$\phi = \sum_{e=0}^{\infty} A_{e0} r^{e} Y_{e0}(\theta, \varphi)$$

$r > a$ $\quad \phi \to 0$ for $r \to \infty \Rightarrow A_{em} = 0$

$$\Rightarrow \phi = \sum_{e=0}^{\infty} B_{e0} \frac{1}{r^{e+1}} Y_{e0}(\theta, \varphi)$$

Potential vanishes on the plane

so $\phi(\theta = \frac{\pi}{2}) = 0$

$$Y_{e0}(\tfrac{\pi}{2}, \varphi) = (1 + (-1)^{e})(-1)^{e/2} c_e$$

$\Rightarrow$ e must be odd

$$\Rightarrow r < a \qquad \phi = \sum_{e=0}^{\infty} A_{e0} (1 - (-1)^{e}) r^{e} Y_{e0}(\theta, \varphi)$$

on z axis $\quad z \to a \quad$ then $\phi \to \frac{q}{a - z} = \sum_n \frac{q z^n}{a^{n+1}}$

$$\Rightarrow A_{e0} = \frac{q}{a^{e+1}} \sqrt{\frac{4\pi}{2e+1}} \qquad \left(Y_{e0}(\theta = 0, \varphi) = \sqrt{\frac{2e+1}{4\pi}} \right)$$

$$\Rightarrow r<a \quad \phi(\theta,\varphi) = \sum_{\ell=0}^{\infty} \frac{q}{a^{\ell+1}} r^{\ell} \sqrt{\frac{4\pi}{2\ell+1}} \left(1-(-1)^{\ell}\right) Y_{\ell 0}(\theta,\varphi)$$

$r>a$ on z axis $z \to a$ $\quad \phi \to \frac{q}{z-a} = \sum_n \frac{q\,a^n}{z^{n+1}}$

$$\Rightarrow A_{\ell 0} = q\,a^{\ell} \sqrt{\frac{4\pi}{2\ell+1}}$$

$$\phi(\theta,\varphi) = \sum_{\ell=0}^{\infty} \frac{q\,a^{\ell}}{r^{\ell+1}} \sqrt{\frac{4\pi}{2\ell+1}} \left(1-(-1)^{\ell}\right) Y_{\ell 0}(\theta,\varphi)$$

The first term is exactly the potential of a charge at $r=a$.

What about the $(-1)^{\ell}$ term

$$(-1)^{\ell}\, Y_{\ell 0}(\theta,\varphi) = Y_{\ell 0}(\pi-\theta,\varphi)$$

$$r<a \quad \phi(r) = \sum_{\ell=0}^{\infty} \frac{(-q)}{a^{\ell+1}} r^{\ell} \sqrt{\frac{4\pi}{2\ell+1}}\, Y_{\ell 0}(\pi-\theta,\varphi)$$

$$r>a \quad \phi(r) = \sum_{\ell=0}^{\infty} \frac{(-q)}{r^{\ell+1}} a^{\ell} \sqrt{\frac{4\pi}{2\ell+1}}\, Y_{\ell 0}(\pi-\theta,\varphi)$$

potential of charge $-q$ at $z=-a$

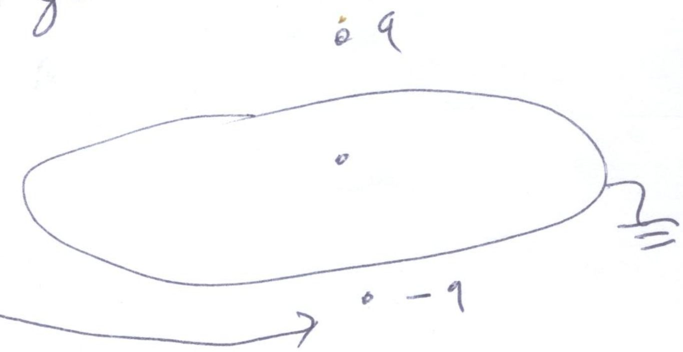
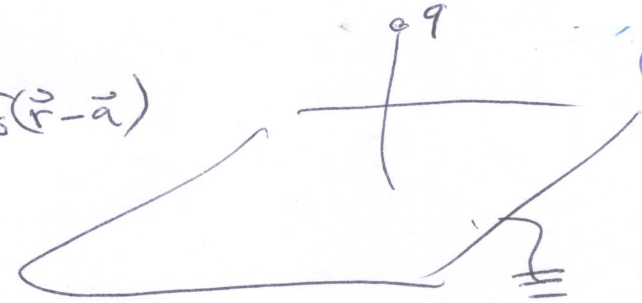


image charge

Alternate Method



$$\nabla^2(\phi_q + \phi_H) = -4\pi\,\delta(\vec r - \vec a)$$

$$\phi_q = \frac{q}{|\vec r - \vec a|}$$

$$\vec\nabla^2 \phi_H = 0 \quad \Rightarrow \quad \phi_H = \sum_\ell \left(A_{\ell 0} r^\ell + \frac{B_{\ell 0}}{r^{\ell+1}}\right) Y_{\ell 0}$$

$$\phi_q \underset{r<a}{=} q \sum a^{-\ell-1} r^\ell \sqrt{\frac{4\pi}{2\ell+1}}\, Y_{\ell 0}$$

$$\underset{r>a}{=} q \sum r^{-\ell-1} a^\ell \sqrt{\frac{4\pi}{2\ell+1}}\, Y_{\ell 0}$$

$\phi_q + \phi_H = 0$ on $z=0$ plane ie for $\theta = \frac{\pi}{2}$

$$Y_{\ell 0}(\theta = \tfrac{\pi}{2}, \varphi) = (1 + (-1)^\ell)\, c_\ell$$

Vanishes for odd ℓ.

$\Rightarrow$ $A_{\ell 0}$ and $B_{\ell 0}$ for odd ℓ are not determined by the boundary conditions

$r<a$ $\quad A_{\ell\,\text{is even}} = -q \dfrac{1}{a^{\ell+1}} \sqrt{\dfrac{4\pi}{2\ell+1}}$

on z axis is $\quad \displaystyle\sum_{\ell=0}^{\infty} -q \frac{r^{2\ell}}{a^{2\ell+1}} \sqrt{\frac{4\pi}{4\ell+1}} \sqrt{\frac{4\ell+1}{4\pi}} = \frac{-q}{a\left(1 - \frac{r^2}{a^2}\right)}$

$$= \frac{-qa}{a^2 - r^2} = \frac{-qa}{2a}\left(\frac{1}{a-r} + \frac{1}{a+r}\right)$$

There can be no singularity at $r = a$

This has to be cancelled by the odd ℓ.

$$\sum_{\ell=0}^{\infty} A_{2\ell+1\,0}\, r^{2\ell+1} \sqrt{\frac{2(2\ell+1)+1}{4\pi}} \quad \Bigg| \quad A_{2\ell+1\,0} = \sqrt{\frac{4\pi}{2(2\ell+1)}} \frac{1}{a^{2\ell+2}}\, q$$

$$\sum_{\ell=0}^{\infty} \frac{q\, r^{2\ell+1}}{a^{2\ell+2}} = \frac{qr}{a^2} \frac{1}{1 - \frac{r^2}{a^2}} = \frac{qr}{a^2 - r^2}$$

combining even and odd gives

$$A_{\ell 0} = -q \frac{(-1)^\ell}{a^{\ell+1}} \sqrt{\frac{4\pi}{2\ell+1}}$$

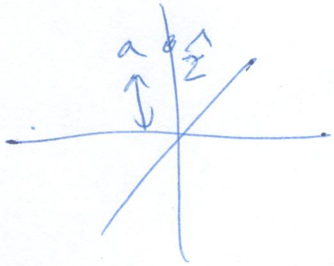
on the z axis this give $\frac{q(r-a)}{a^2-r^2} = \frac{-q}{a+r}$

this is the potential of charge $-q$ at $z=-a$

$$\Rightarrow \quad \Phi^{H}_{r<a} = \sum_{\ell=0}^{\infty} (-q) \frac{(-1)^\ell}{a^{\ell+1}} r^\ell \sqrt{\frac{4\pi}{2\ell+1}}\, Y_{\ell 0}(\theta, \varphi)$$

The analysis for $r > a$ is completely analogous.

Lecture #6



$$\phi(r<a) = \sum_{l=0}^{\infty} \frac{q\, r^l}{a^{l+1}}$$

$$\phi(r>a) = \sum_{l=0}^{\infty} \frac{q\, a^l}{r^{l+1}}$$

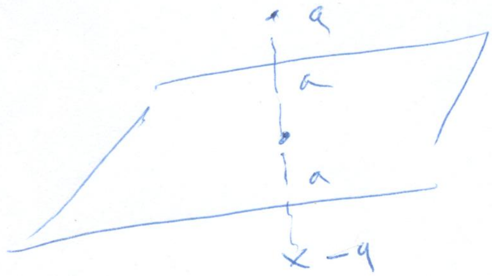
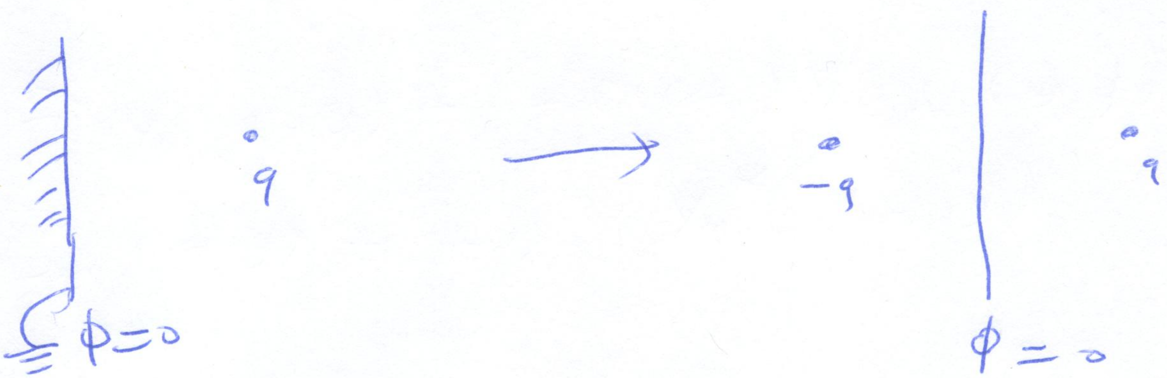


image char
out from expa
Spherical har

Today:
- Uniqueness theore
- general proper
- image charges

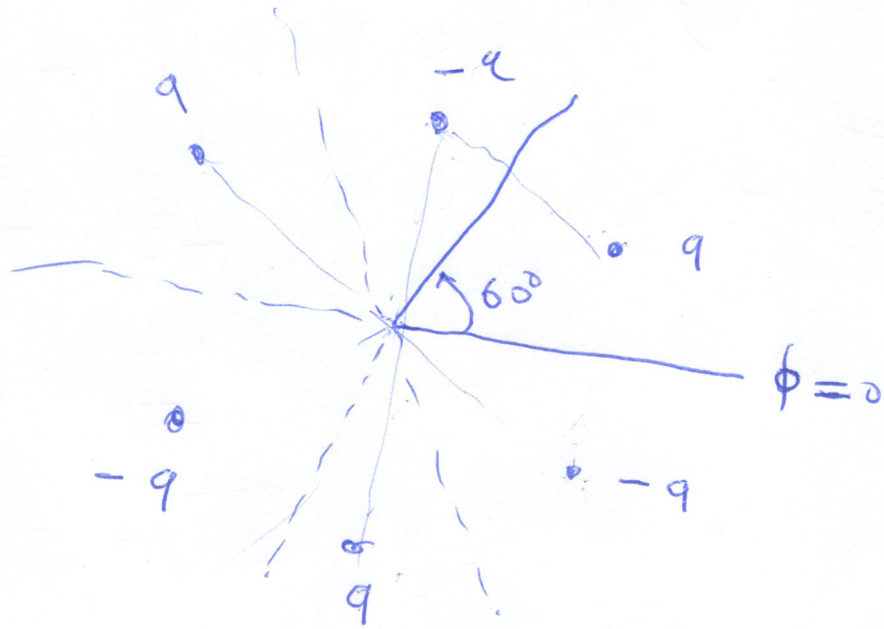
This problem can be solved
by image charges.
The trick is to find other charges
that give the same boundary condition



$\Rightarrow$ Solution on the right of the plane is the same, not on the left

$$\phi(\vec{r}) = \frac{q}{|\vec{r} - \vec{a}|} - \frac{q}{|\vec{r} + \vec{a}|}$$

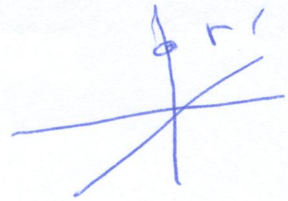
2nd Example of image charges



Expansion of $\frac{1}{|\vec{r}-\vec{r}\,'|}$ in spherical harmonics

choose $|\vec{r}| > |r'|$

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \sum \frac{1}{r^{\ell+1}} A_{\ell m} P_\ell \, Y_{\ell m}(\theta, \varphi)$$



choose $\vec{r}\,'$ on z axis

on z axis: $\frac{1}{|r - r'|} = \sum_{k=0}^{\infty} \frac{r'^k}{r^{k+1}}$

$\cancel{\Rightarrow}$ no dependence on $\varphi \Rightarrow m = 0$

$$\Rightarrow A_{\ell 0} = \frac{r'^\ell}{Y_{\ell 0}(0,\varphi)} = r'^\ell \left(\frac{4\pi}{2\ell+1}\right)^{1/2} Y_{\ell 0}(0,\varphi)$$

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \sum \frac{1}{r^{\ell+1}} r'^\ell \frac{4\pi}{2\ell+1} Y_{\ell 0}(0, \varphi)\, Y_{\ell 0}(\theta, \varphi)$$

$$Y_{\ell 0}(\theta', \varphi)\, Y_{\ell 0}(\theta, \varphi)$$

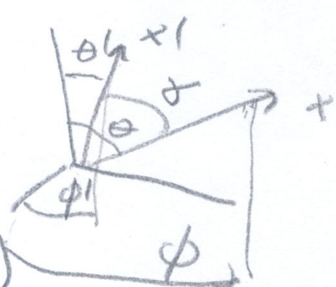
only depends on angle between $\vec{r}$ and $\vec{r}\,'$

$$Y_{\ell 0}(\theta,\varphi) = \sqrt{\frac{2\ell+1}{4\pi}}\, P_\ell(\cos\theta)$$

$$Y_{\ell 0}(0,0) = \sqrt{\frac{2\ell+1}{4\pi}}$$

addition theorem

$$P_\ell(\cos\gamma) = \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{\ell} Y^*_{\ell m}(\theta', \varphi') \times Y_{\ell m}(\theta, \varphi)$$

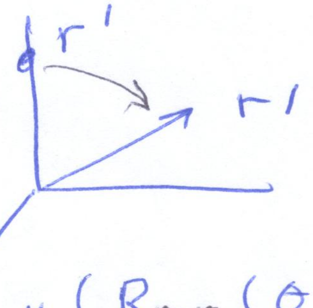


$$\Rightarrow \frac{1}{|\vec{r}-\vec{r}\,'|} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{4\pi}{(2\ell+1)} \frac{r'^\ell}{r^{\ell+1}} Y^*_{\ell m}(\theta', \varphi')\, Y_{\ell m}(\theta, \varphi)$$

$$\frac{1}{|\vec{r}-\vec{r}'|} = \sum_{\ell} \frac{r'^{\ell}}{r^{\ell+1}} \sqrt{\frac{4\pi}{2\ell+1}}\, Y_{\ell 0}(\theta, \varphi)$$

on z-axis

now rotate $r' \to (\theta', \varphi')$



then $Y_{\ell m}(\vec{r}) = \sum_{m''} \mathcal{D}^{\ell}_{m'' m}\, Y_{\ell m''}(\underbrace{R_{\theta' \varphi'}(\theta, \varphi)}_{\text{new } \theta, \varphi \text{ coordinates of } \vec{r}})$

$m \to 0$, $m \to 0$

Wigner $\mathcal{D}$ matrices

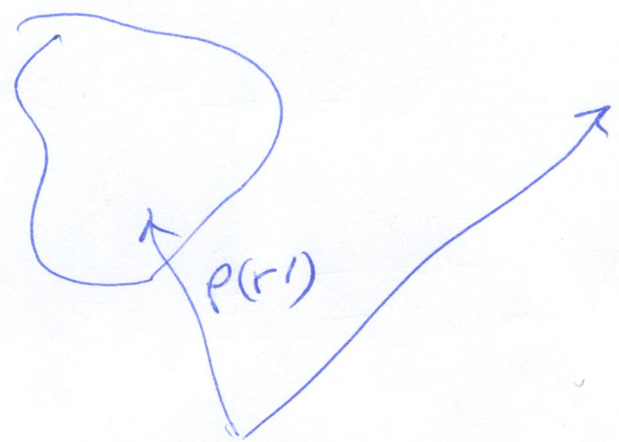
$$\mathcal{D}^{\ell}_{m''0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y^{*}_{\ell m''}(\theta', \varphi')$$

$$\Rightarrow \frac{1}{|\vec{r}-\vec{r}'|} = \sum_{\ell, m''} \frac{r'^{\ell}}{r^{\ell+1}} \frac{4\pi}{2\ell+1} Y^{*}_{\ell m''}(\theta', \varphi') Y_{\ell m''}(\theta, \varphi)$$

note that $r > r'$

Multipole expansion

a)



$$\phi(r) = \int d^3r' \, \rho(r') \frac{1}{|\vec{r}\,' - \vec{r}\,|}$$

$$= \int d^3r' \, \rho(r') \sum_{\ell m} \frac{r'^{\ell}}{r^{\ell+1}} \frac{4\pi}{2\ell+1} \rho(r') Y^*_{\ell m}(\theta', \varphi')$$

$$\times Y_{\ell m}(\theta, \varphi)$$

multipole moment

$$Q_{\ell m} = \int d^3r' \, r'^{\ell} \rho(\vec{r}\,') \, Y^*_{\ell m}(\theta' \varphi')$$

$$\Rightarrow \phi(r) = \sum_{\ell=0}^{\infty} \frac{4\pi}{2\ell+1} \frac{1}{r^{\ell+1}} \sum_{m=-\ell}^{\ell} Y_{\ell m}(\theta, \varphi) \, Q_{\ell m}$$

$\ell = 0$ monopole

$\ell = 1$ dipole

$\ell = 2$ quadrapole

Lecture #7

$$\frac{1}{|\vec{r}-\vec{r}'|} = \sum_{lm} \frac{r'^l}{r^{l+1}} \frac{4\pi}{2l+1} Y^*_{lm}(\theta',\varphi) Y_{lm}(\theta,\varphi)$$

$r > r'$

$r < r'$ then $r' \leftrightarrow r$

multipole expansion

$$Q_{lm} = \int d^3r' \, r'^l \rho(r') Y^*_{lm}(\theta',\varphi)$$

$$\Phi(r) = \sum_{l=0}^{\infty} \frac{4\pi}{2l+1} \frac{1}{r^{l+1}} \sum_{m=-l}^{l} Y_{lm}(\theta,\varphi) Q_{lm}$$

$l=0$ monopole
$l=1$ dipole
$l=2$ quadrupole

Today - image charges
- Dipole moment
- Electrostatics of conductors

2 d Gauss theorem $\int_D \vec{\nabla}\cdot\vec{A}\, d^2a = \int_{\partial D} \vec{A}\cdot\hat{n}\, ds$

Stokes theorem $\int_D \vec{\nabla}\times\vec{A}\, d^2a = \int_{\partial D} \vec{A}\cdot d\vec{s}$

2d E field $\vec{E} = q\frac{\vec{x}}{r^2} + q\frac{\vec{y}}{r^2}$

$$\int_D \vec{\nabla}\cdot\vec{E} = \int_{\partial D} q\left(\frac{\vec{x}\cdot\hat{x}}{r^2} + \frac{\vec{y}\cdot\hat{y}}{r^2}\right) ds = \frac{q}{r} 2\pi r = 2\pi q$$

b) <u>Dipole moment</u>

$$\vec{p} = \int d^3x' \, \rho(x') \, \vec{x}'$$

$$Q_{10} = \sqrt{\frac{3}{4\pi}} \int z' \rho(x') \, d^3x' = \sqrt{\frac{3}{4\pi}} \, p_z$$

$$Q_{11} = -\sqrt{\frac{3}{8\pi}} \int (x' - iy') \, \rho(x') \, d^3x'$$

$$= -\sqrt{\frac{3}{8\pi}} \, (p_x - i p_y)$$

$$Q_{1-1} = - Q_{11}^{*} = \sqrt{\frac{3}{8\pi}} \, (p_x + i p_y)$$

The potential of a dipole is given by

$$\phi(r) = \frac{4\pi}{3} \frac{1}{r^2} \Big(Y_{11} \left(-\sqrt{\frac{3}{8\pi}}\right) (p_x - i p_y)$$

$$\overset{\nearrow}{2l+1} \qquad + Y_{10} \sqrt{\frac{3}{4\pi}} \, p_z$$

$$+ Y_{1-1} \sqrt{\frac{3}{8\pi}} \, (p_x + i p_y) \Big)$$

$$Y_{11} = -\sqrt{\frac{3}{8\pi}} (x + iy)/r \qquad Y_{10} = \sqrt{\frac{3}{4\pi}} \frac{z}{r}$$

$$Y_{1-1} = \sqrt{\frac{3}{8\pi}} \frac{(x - iy)}{r}$$

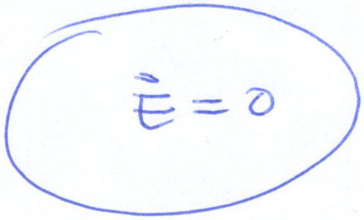
$$\phi(r) = \frac{4\pi}{3} \frac{1}{r^3} \Big((x+iy)(p_x - i p_y) \frac{3}{8\pi} + z \, p_z \frac{3}{4\pi}$$

$$+ (x - iy)(p_x + i p_y) \frac{3}{8\pi} \Big)$$

$$= \frac{1}{r^3} (x p_x + y p_y + z p_z) = \frac{1}{r^3} (\vec{r} \cdot \vec{p})$$

Electrostatics of conductors

a)



$\vec{E} = 0 \Rightarrow \vec{\nabla} \vec{E} = 0 \Rightarrow \rho = 0$

$\Rightarrow$ we can only have a surface charge density on a conductor.

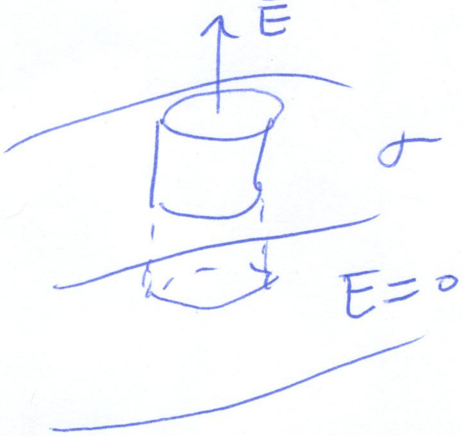
$\vec{E} = 0 \quad \vec{E} = \vec{\nabla} \phi \Rightarrow$

- conductor is an equipotential surface

$$\phi_{12} = -\int_1^2 \vec{E} \cdot ds$$

- $\vec{E} \perp$ surface, otherwise the electrons will rearrange

- Surface charge density



$$\oint \vec{E} \cdot \vec{n} \, da = 4\pi \sigma A$$

$$\Rightarrow E_\perp A = 4\pi \sigma A$$

$$E_\perp = 4\pi \sigma$$

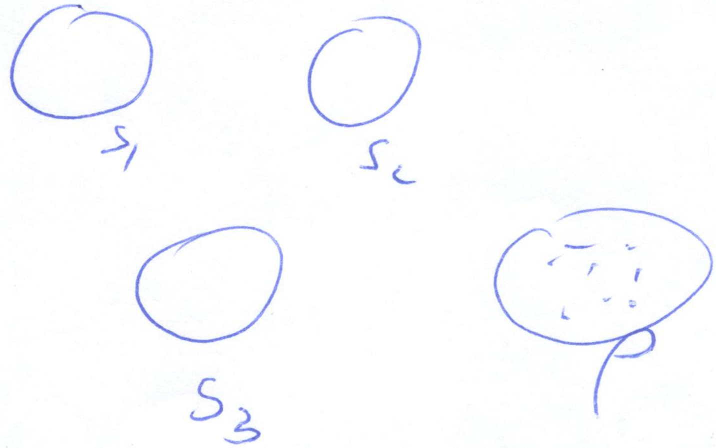
Lecture # 8

Today: Uniqueness Theorem
Green's Function
Application

Uniqueness theorem

give ρ, conducting surfaces S_i with charge Q_i or potential ϕ_i, then the electric field is determined uniquely

Proof



Suppose we have two different potentials

$$\vec{E}_1 = -\vec{\nabla}\varphi_1 \qquad E_2 = -\vec{\nabla}\varphi_2$$

then $$I = \int_{V = \mathbb{R}^3/\cup(S_i)} d^3r\, (\vec{\nabla}\varphi_1 - \vec{\nabla}\varphi_2)^2 \neq 0$$

apply Gauss theorem to $\vec{\nabla}\left[(\varphi_1^i - \varphi_2^i)\vec{\nabla}(\varphi_1^i - \varphi_2^i)\right]$

$$= \int_V d^3r\, \vec{\nabla}(\varphi_1 - \varphi_2)\,\vec{\nabla}(\varphi_1 - \varphi_2)$$

$$= \sum_i \int_{S_i} da\, \vec{n}\,(\varphi_1^i - \varphi_2^i)\,\vec{\nabla}(\varphi_1^i - \varphi_2^i) - \int_V d^3r\,(\varphi_1 - \varphi_2) \times \underbrace{\vec{\nabla}^2(\varphi_1 - \varphi_2)}_{=0}$$

φ_k^i is constant on S_i

$\vec{\nabla}^2\varphi_1 = -4\pi\rho \quad \vec{\nabla}^2\varphi_2 = -4\pi\rho$

$$= \sum_i \int_{S_i} (\varphi_1^i - \varphi_2^i)\, \vec{n}\cdot(\vec{E}_{1i} - \vec{E}_{2i})$$

$$= \sum_i (\varphi_1^i - \varphi_2^i)(Q_1^i - Q_2^i) \Rightarrow \text{qed}$$

Green's function

Definition $\vec{\nabla}^2_{x'} G(\vec{x}, \vec{x}') = -4\pi\, \delta(\vec{x} - x')$

So $G(\vec{x}, x') = \dfrac{1}{|\vec{x} - \vec{x}'|} + F(\vec{x}, \vec{x}')$

with $\vec{\nabla}^2_{x'} F(\vec{x}, \vec{x}') = 0$

$\vec{F}$ is determined by the boundary conditions

For a single charge in vacuum

$$G(\vec{x}, \vec{x}') = \frac{1}{|\vec{x} - x'|}$$

is potential at x from unit charge at x'

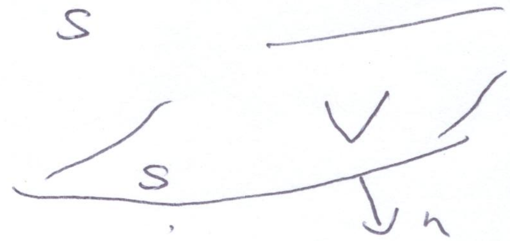
Green's function is symmetric

$$\int d^3r \; G(\vec{r}, \vec{r}_1)\, \underbrace{\vec{\nabla}^2_{r}\, G(\vec{r}, r_2)}_{-4\pi\,\delta(\vec{r} - \vec{r}_2)} = \int d^3r \; \underbrace{\vec{\nabla}^2_r G(\vec{r}, \vec{r}_1)}_{-4\pi\,\delta(\vec{r} - r_1)}\, G(\vec{r}, \vec{r}_2)$$

$$\Rightarrow -4\pi\, G(\vec{r}_2, \vec{r}_1) = -4\pi\, G(\vec{r}_1, \vec{r}_2)$$

Application of Green's Function

Gauss: $$\int_V \vec{\nabla}\cdot\vec{A}\, d^3x = \oint_S \vec{A}\cdot\vec{n}\, da$$



choose $\vec{A} = \phi\,\vec{\nabla}\psi$

arbitrary

Then $$\vec{\nabla}\vec{A} = \phi\,\vec{\nabla}^2\psi + \vec{\nabla}\phi\,\vec{\nabla}\psi$$
$$\vec{A}\cdot\vec{n} = \phi\,\vec{\nabla}\psi\,\vec{n} \equiv \phi\,\partial_n\psi$$

$$\Rightarrow \int_V (\phi\,\vec{\nabla}^2\psi + \vec{\nabla}\phi\,\vec{\nabla}\psi)\,d^3x = \int_S \phi\,\vec{\nabla}\psi\cdot\vec{n}\,da$$

Green I

subtract same eq with ϕ and ψ interchanged

Green II: $$\int_V (\phi\,\vec{\nabla}^2\psi - \psi\,\vec{\nabla}^2\phi)\,d^3x' = \oint_S \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right)da'$$

choose ϕ a potential $\vec{\nabla}^2\phi = -4\pi\rho$

and ψ a Green's function $\psi = G(x', x)$

$$\nabla^2_{x'}\, G(\vec{x}', \vec{x}) = -4\pi\,\delta^3(x', x)$$

$$-4\pi\phi(x) + 4\pi\int \rho(x')\,G(x', x)\,d^3x'$$
$$= \oint_S \phi(x')\left(\vec{\nabla}_{x'}G(x', x) - G(x', x)\,\vec{\nabla}\phi\right)\cdot\vec{n}\,da$$

choose Green's function such that $G(x', x) = 0$ if $x' \in S$

$$\Rightarrow \phi(x) = \int \rho(x')\,G(x', x)\,d^3x' - \oint_S \phi(x')\,\vec{\nabla}_{x'}G(x', x)\cdot\vec{n}\,da$$

Lecture # 9

Green's Function $\vec{\nabla}^2_{x'} G(\vec{x}, x') = -4\pi \delta^3(x - x')$

$$G(x, y) = G(y, x)$$

$$\phi(x) = \int \rho(x') G(x', x)\, d^3x - \oint_S \phi(x') \vec{\nabla}_{x'} G(x', x) \cdot \hat{n}\, da$$

Today
- Energy
- stress tensor
- Example

4. Energy and stress in an electrostatic Field

1) Energy

Work done to bring a small charge dq_i from ∞ to r_i

$$\delta W_i = (\phi(r_i) - \phi(\infty))\,\delta q_i$$
$$= -\int_\infty^{r_i} \vec{E}\cdot d\vec{s}\;\delta q_i$$

total work for many charges broaght from infinity

$$\delta W = \sum_i \delta W_i = \sum_i \phi(r_i)\,\delta q_i$$
$$= \int d^3r\;\phi(r)\,\delta\rho(r)$$

$$\vec{\nabla}\vec{E} = -4\pi\rho \qquad \Rightarrow \qquad \delta\rho = \frac{1}{4\pi}\vec{\nabla}\,\delta\vec{E}$$

$$\Rightarrow \delta W = \int d^3r\;\phi(r)\,\frac{1}{4\pi}\vec{\nabla}\,\delta E$$
$$\underset{\text{partial integration}}{=} -\int d^3r\;\vec{\nabla}\phi\,\frac{1}{4\pi}\,\delta E$$

we integrate over all space $\Rightarrow$ surface term vanishes

$$\Rightarrow \delta W = \frac{1}{4\pi}\int d^3r\;\vec{E}\cdot\delta E$$
$$= \frac{1}{8\pi}\int d^3r\;\delta\vec{E}^2$$

Work done to change the Field strength from $E_0 \to E_f$

$$W = \frac{1}{8\pi}\int d^3r\,(E_f^2 - E_0^2)$$

$\vec{E}$ is changed by taking charges from ∞ to r_i

$\Rightarrow$ energy density is $\dfrac{\vec{E}^2}{8\pi}$

b) <u>Stress tensor</u>

Force on charges inside S

$$F_i = \int_V d^3r\, \rho(\vec{r})\, E_i(\vec{r})$$

$\leftarrow$ i'th component of $\vec{E}$ $\quad i = 1,2,3 \quad \partial V = S$

$$= \frac{1}{4\pi}\int_V d^3r\, \vec{\nabla}\cdot\vec{E}\, E_i$$

$$= \frac{1}{4\pi}\int_V d^3r\left(\partial_j (E_j E_i) - E_j \partial_j E_i\right)$$

$$\vec{\nabla}\times\vec{E} = 0 \Rightarrow \varepsilon_{ijk}(\partial_j E_k - \partial_k E_j) = 0$$

$$\Rightarrow \partial_j E_k - \partial_k E_j = 0 \quad k \neq j$$

and trivially $= 0$ for $k = j$

$$\Rightarrow E_j \partial_j E_i = E_j \partial_i E_j + \underbrace{E_j \partial_j E_i - E_j \partial_i E_j}_{=0}$$

$$= \frac{1}{2}\partial_i E_j^2$$

$$\Rightarrow F_i = \frac{1}{4\pi}\int_V d^3r\, \partial_j \left(E_j E_i - \delta_{ij}\frac{\vec{E}^2}{2}\right)$$

$$\overset{\text{Gauss}}{=} \int_S dS_j\, T_{ji}$$

Maxwell stress tensor $\quad T_{ji} = \frac{1}{4\pi}\left(E_j E_i - \delta_{ij}\frac{\vec{E}^2}{2}\right)$

Example

$\vec{E} \perp$ surface at surface

$$\Rightarrow \vec{E} = (0, 0, E_z)$$

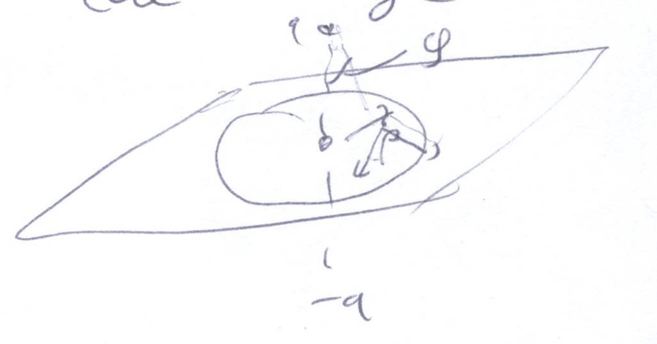
$$T_{ji}\Big|_s = \begin{pmatrix} -\frac{1}{2}E_z^2 & & \\ & -\frac{1}{2}E_z^2 & \\ & & \frac{1}{2}E_z^2 \end{pmatrix} \frac{1}{4\pi}$$

$$F_i = \int ds_i T_{ji}$$

$$= \int \underset{\uparrow}{\hat{n}}\, ds\, T_{33} = -\int_{ds} \frac{1}{8\pi} E_z^2$$

normal
to the outside from the point of view of the charge

E is the field due to the charge and image charge



$$\vec{E} = -\frac{q}{(a^2+\rho^2)} 2\cos\varphi$$

$$= -\frac{2qa}{(a^2+\rho^2)^{3/2}} \quad \text{at } \varphi = 0$$

$$\vec{F} = -\int ds \frac{1}{8\pi} E_z^2 = -\int ds \frac{1}{8\pi} \frac{4q^2a^2}{(a^2+\rho^2)^3}$$

$$\int \rho\, d\rho\, d\varphi$$

$$= -\frac{2\pi}{8\pi} 4 \int_0^\infty \rho\, d\rho \frac{q^2a^2}{(a^2+\rho^2)^3} = -\frac{q^2}{a^2}\int_0^\infty \frac{\rho\, d\rho}{(1+\rho^2)^3}$$

$$= -\frac{q^2}{4a^2}$$ correct

Energy in terms of charges

$$W_E = \frac{1}{8\pi}\int_V \vec{E}^2 d^3r$$

$$= \frac{1}{8\pi}\int_V \vec{\nabla}\phi\,\vec{\nabla}\phi\, d^3r$$

$$= -\frac{1}{8\pi}\int_V \phi\,\vec{\nabla}^2\phi\, d^3r$$

$$= \frac{1}{2}\int d^3r\ \phi(r)\,\rho(r)$$

$$= \frac{1}{2}\int d^3r\, d^3r' \frac{\rho(r)\,\rho(r')}{|\vec{r}-\vec{r}'|}$$

↑ factor $\frac{1}{2}$ because this is the work energy to bring charges from infinity.

For a collection of point charge

$$W = \frac{1}{2}\sum_{i\neq j}\frac{q_i q_j}{|\vec{r}_i-\vec{r}_j|}$$

the energy is always because of the field from another charge.

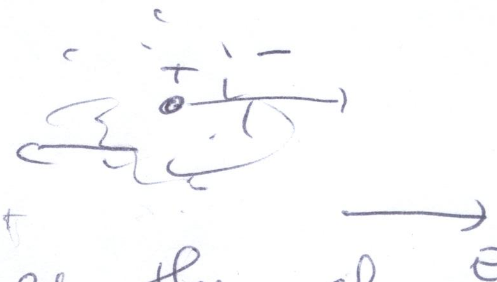
we see later that the energy of a magnetic field is $W_B = \frac{1}{8\pi}\int d^3r\ \vec{B}^2$

$$T + W_E + W_B = \text{constant}$$

↑ Kinetic energy

5) Electrostatics in matter

- in conductors electrons can move freely
- in dielectric the electrons are bound to the atoms



in electric field the charge distribution of the electrons gets displaced wrt the nucleus and the atom gets a dipole moment.

<u>model</u> This effect can be approximated by a dipole density $\vec p(r)$

This gives the potential

$$\phi(r) = \sum_i \frac{\vec p_i (\vec r - \vec r_i)}{|(\vec r - r_i)|^3}$$

$$= -\sum p_i \vec\nabla_{\vec r} \frac{1}{|\vec r - \vec r_i|}$$

$$= -\int d^3r' \, \vec p(r') \underbrace{\vec\nabla_{\vec r}}_{= -\vec\nabla_{r'}} \frac{1}{|\vec r - \vec r'|}$$

$$= \int d^3r' \left(\nabla_{r'}\left(p(r') \frac{1}{|\vec r - \vec r'|}\right) - \frac{(\nabla_{r'} p(r'))}{|r-r'|} \right)$$

$$= \int_{S=\partial V} \frac{\vec{P}(r')\cdot d\vec{a}}{|r-r'|} + \int d^3r' \, \frac{\left(-\vec{\nabla}_{\vec{r}'} P(\vec{r}')\right)}{|r-r'|}$$

potential of surface charge density on S

$\sigma = \vec{P}\cdot\vec{n}$

induced charge density

$\rho_b = -\vec{\nabla}_{r'} P(r')$

- if the dipole density is constant then $\vec{\nabla}_{r'}\vec{P}(r') = 0$ and we only have a surface charge density

effect of induced charge density

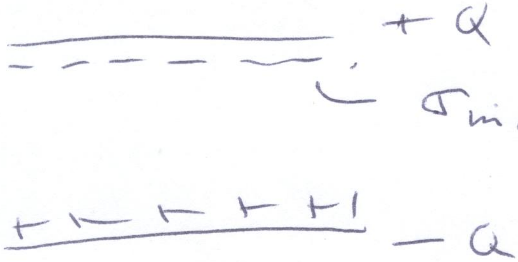
$$\vec{\nabla}\vec{E} = 4\pi(\rho + \rho_{ind})$$
$$= 4\pi(\rho - \vec{\nabla}\vec{P})$$

$$\Rightarrow \vec{\nabla}(\underbrace{\vec{E} + 4\pi\vec{P}}_{\equiv \vec{D}}) = 4\pi\rho$$

free charges

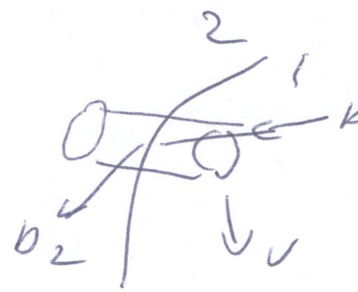
For linear media we have $\vec{D} = \varepsilon\vec{E}$

For a surface charge density we have.



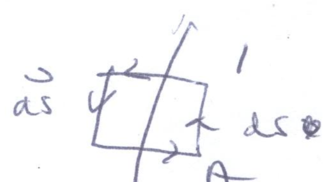
$$\vec{E} = 4\pi(\sigma - \sigma_{ind}) \cdot \vec{n}$$
$$= 4\pi(\sigma - (\vec{P}\cdot\vec{n}))\vec{n}$$
$$\Rightarrow (\vec{E} + 4\pi\vec{P}) = 4\pi\sigma\vec{n}$$

5b) <u>Boundary conditions in a medium</u>



$$\int_{\partial V} \vec{D}\cdot\vec{n}\,da = 4\pi\sigma A$$
$$\Rightarrow D_{2n} - D_{1n} = 4\pi\sigma$$

$$\vec{\nabla}\vec{E} = 0$$



$$\int_{\partial A} \vec{E}\cdot\vec{ds} = \int \vec{\nabla}\times\vec{E}\cdot\vec{dA} = 0$$
$$E^1_{tang} - E^2_{tang} = 0$$

Electrostatic energy in a medium

We do this again by building up a charge density

Work done

$$\delta W = \sum_i \delta q_i \, \phi(\vec{r}_i)$$

$$= \int d^3r \, \delta\rho(\vec{r}) \, \phi(\vec{r})$$

$$= \int d^3r \, \frac{\vec{\nabla}\delta\vec{D}}{4\pi} \, \phi(r)$$

$$= -\int d^3r \, \frac{\delta\vec{D}}{4\pi} \, \vec{\nabla}\phi$$

$$= \int d^3r \, \frac{\delta\vec{D}\,\vec{E}}{4\pi}$$

for a linear medium $\qquad \vec{D}_i = \varepsilon_{ij} E_j$

$$= \frac{1}{4\pi}\int d^3r \; \varepsilon_{ij} \, \delta E_j \, E_i$$

$$\varepsilon_{ij} = \varepsilon^S_{ij} + \varepsilon^A_{ij} \qquad \varepsilon^S_{ij} = \varepsilon_{ji}$$

$$\varepsilon^A_{ij} = -\varepsilon_{ji}$$

$$\frac{1}{4\pi}\int d^3r \; \varepsilon^S_{ij} \, \delta E_j \, E_i = \frac{1}{4\pi}\int d^3r \; \varepsilon_{ij} \frac{1}{2} \delta(E_i E_j)$$

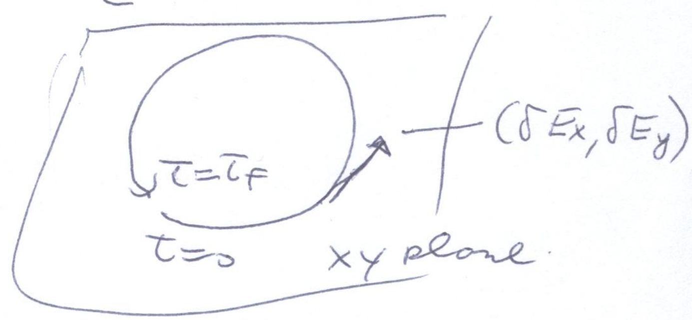
$$= \frac{1}{8\pi}\int d^3r \; \delta(\vec{D}^S \cdot \vec{E})$$

$$\Rightarrow W = \frac{1}{8\pi}\int d^3r \; \vec{D}^S \cdot \vec{E}$$

For the anti-symmetric part of the dielectric constant we obtain

$$\delta W = \frac{1}{4\pi}\int d^3r\, \varepsilon^A_{ij}\, \frac{1}{2}\left(\delta E_j \cdot E_i - \delta E_i \cdot E_j\right)$$

We now let the electric Field depend on a parameter τ



$$\vec{E} \to \vec{E}(\tau)$$

and calculate the work after $\vec{E}$ returns again to the same value

$$\oint \delta W\, d\tau = \frac{1}{4\pi}\int d^3r\, \frac{\varepsilon^A_{ij}}{2} \oint d\tau \left(\delta E_j E_i - \delta E_i E_j\right)$$

we consider $\vec{E}$ in the xy plane.

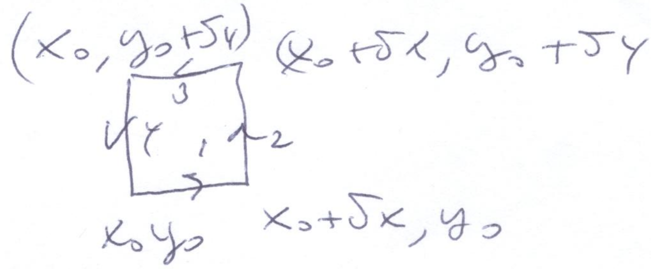
$$= \frac{1}{4\pi}\int d^3r\, \frac{\varepsilon^A_{xy}}{2} \oint d\tau \left(\underbrace{\delta E_y}_{=\frac{dE_y}{d\tau}} E_x - \underbrace{\delta E_x}_{=\frac{dE_x}{d\tau}} E_y\right)$$

$$\oint \left(E_x\, dE_y - E_y\, dE_x\right)$$

This is in E_x, E_y space

Let us look how this work for x, y : $\frac{1}{2}\oint (x\, dy - y\, dx$

We consider an infinitesimal loop



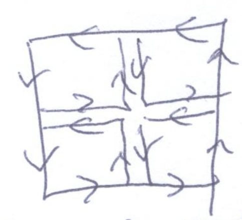
$$\frac{1}{2}\int(x\,dy - y\,dy)$$

$$\frac{1}{2}\Big(-\underset{①}{y_0\,\delta x} + \underset{②}{(x_0+\delta x)\,\delta y}$$

$$(-)(-)\underset{③}{(y_0+\delta y)}\,\delta x \quad - \underset{④}{x_0\,\delta y}$$

$$= \frac{1}{2}\; 2\,\delta x\,\delta y \quad = \quad \text{area of loop}$$

For bigger loop



integral over the inner loops cancels.

$\Rightarrow$ $\delta W \neq 0$ when $\vec{E}$ returns to its original value.

$\Rightarrow$ Σ^A_{ij} describes absorbtion