Lecture #4

Gauss $\vec{\nabla} E = 4\pi\rho$

Poisson $\vec{\nabla}^2 \phi = -4\pi\rho$

Laplace $\vec{\nabla}^2 \phi = 0$

Today: Uniqueness

Solution of Laplace equation

Integral form of Gauss law

$$\int_V dV\, \vec{\nabla}\cdot\vec{E} = \int dV\, 4\pi\rho$$

$$\parallel$$

$$\int_{\partial V} da\, \vec{n}\cdot\vec{E}$$

e) Poisson law

$$\left.\begin{aligned}\vec{\nabla}\vec{E} &= 4\pi\rho \\ \vec{E} &= -\vec{\nabla}\phi\end{aligned}\right\} \Rightarrow \vec{\nabla}^2\phi = -4\pi\rho$$

Poisson equatia

$\rho = 0$ $\quad \vec{\nabla}^2\phi = 0$ Laplace eq

solutions are called harmonic function.

f) Uniqueness

A vector field that satisfies

$\vec{\nabla}\vec{E} = 0$ $\quad \vec{\nabla}\times\vec{E}$

and vanishes at least as $\frac{1}{r^2}$ for $r\to\infty$

vanishes everywhere

This follows from liouville's theoreme:

if f is a harmonic function on $\mathbb{R}^n$

and f is bounded from above or below

then f is constant

Reason $\vec{E} = -\vec{\nabla}\phi$

$$\vec{\nabla}^2\phi = 0$$

$\vec{E} \sim \frac{1}{r^2}$ for $r \to \infty \Rightarrow \phi \sim \frac{1}{r}$ for $r \to \infty$

$\vec{\nabla}.\vec{E} = 0 \Rightarrow \exists$ no singularities

$\Rightarrow \phi$ is bounded $\Rightarrow \phi =$ constant

$\Rightarrow \vec{E} = 0$

Consequence: uniqueness theorem

$$\left.\begin{aligned} \vec{\nabla}.\vec{E}_1 &= 4\pi\rho_1 \\ \vec{\nabla}.\vec{E}_2 &= 4\pi\rho_2 \end{aligned}\right\} \Rightarrow \vec{\nabla}(\vec{E}_1 - E_2) = 0$$

statics $\Rightarrow \vec{\nabla}\times(\vec{E}_1 - \vec{E}_2) = 0$

$\Rightarrow E_1 - E_2 = 0$

Solutions of the Laplace eq

$$z = r\cos\theta$$
$$x = r\sin\theta\cos\varphi$$
$$y = r\sin\theta\sin\varphi$$

$$\vec{\nabla}^2\phi = 0$$

spherical coordinates

$$\vec{\nabla}^2\phi = \frac{1}{r}\partial_r^2(r\phi) + \frac{1}{r^2\sin\theta}\partial_\theta\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2}$$

$$\phi = \frac{u(r)}{r}P(\theta)Q(\varphi)$$

sepration of variables

$$\nabla^2\phi = \frac{PQ}{r}\partial_r^2 u + \frac{uQ}{rr^2\sin\theta}\partial_\theta(\sin\theta\,\partial_\theta P) \underline{g}$$

$$+ \frac{uP}{rr^2\sin^2\theta}\frac{\partial^2 Q}{\partial\varphi^2} = 0$$

divide by uPQ

$$\frac{u^{-1}}{r}\partial_r^2 u + \frac{P^{-1}}{rr^2\sin\theta}\partial_\theta(\sin\theta\,\partial_\theta P)$$

$$+ \frac{Q^{-1}}{rr^2\sin^2\theta}\frac{\partial^2 Q}{\partial\varphi^2} \Rightarrow$$

$\times r^3$

$$r^2 u^{-1}\partial_r^2 u + \frac{P^{-1}}{\sin\theta}\partial_\theta(\sin\theta\,\partial_\theta P)$$

$$+ \frac{Q^{-1}}{\sin^2\theta}\frac{\partial^2 Q}{\partial\varphi^2} = 0$$

only depends on r

only depends on θ, φ

$\Rightarrow$ both should be constant because the equation is valid for all r, θ, φ

$$r^2 u^{-1}\partial_r^2 u = c \Rightarrow \partial_r^2 u = c\frac{u}{r^2}$$

$$u = \alpha r^{\ell+1} \Rightarrow \ell(\ell+1) r^{\ell-2} = c\, r^{\ell-2}$$

$$\Rightarrow c = \ell(\ell+1)$$

$\frac{r_0}{V}$

then

$$\ell(\ell+1) \quad \sin^2\theta \quad + \sin\theta \, P^{-1} \partial_\theta (\sin\theta \, \partial_\theta P)$$

$$+ Q^{-1} \frac{\partial Q^2}{\partial \varphi^2} = 0$$

$$\Rightarrow \ell(\ell+1) \sin^2\theta + \sin\theta \, P^{-1} \partial_\theta (\sin\theta \, \partial_\theta P)$$

$$= -\left(\frac{\partial Q^2}{\partial \varphi^2}\right) Q^{-1}$$

function of θ function of φ

equal $\forall \, \theta, \varphi \Rightarrow$ they have to be constant

$$\Rightarrow Q^{-1} \frac{\partial^2 Q^2}{\partial \varphi^2} = -m^2 \Rightarrow Q = e^{\pm i m \varphi}$$

$$\Rightarrow \sin\theta \, \partial_\theta (\sin\theta \, \partial_\theta P) = \ell(\ell+1)\sin^2\theta \, P - m^2 P$$

This equation is known as the Legendre equation. $P = P(\cos\theta)$

The solutions are given by the associated Legendre polynomials

The combination PQ is given by $PQ = c' P_{\ell m}(\cos\theta)\, e^{im\varphi} \equiv c' Y_{\ell m}(\theta,\varphi)$

spherical harmonics

The constant c' is fixed by the normalization sum rule

$$\sum_{m=-\ell}^{\ell} Y_{\ell m}(\theta,\varphi)\, Y^*_{\ell m}(\theta,\varphi) = \frac{2\ell+1}{4\pi}$$

$$Y_{00} = \frac{1}{\sqrt{4\pi}}$$

$$Y_{11} = -\sqrt{\frac{3}{8\pi}} \sin\theta\, e^{i\phi} = -\sqrt{\frac{3}{8\pi}}\, \frac{x+iy}{r}$$

$$Y_{10} = \sqrt{\frac{3}{4\pi}} \cos\theta = \sqrt{\frac{3}{4\pi}}\, \frac{z}{r}$$

$$Y_{1-1} = \sqrt{\frac{3}{8\pi}} \sin\theta\, e^{-i\varphi} = \sqrt{\frac{3}{8\pi}}\, \frac{x-iy}{r}$$

We have found an infinite set of harmonic functions

$$\nabla^2 r^\ell \, Y_{\ell m}(\theta,\varphi) = 0$$

$$\ell(\ell+1) = \quad (-\ell-1)((-\ell-1)+1)$$

$$\Rightarrow u(r) = \frac{1}{r^\ell} \text{ is also solution}$$

$$\Rightarrow \quad \vec{\nabla}^2 \frac{1}{r^{\ell+1}} Y_{\ell m}(\theta, \varphi) = 0$$

Most general solution of the Laplace eq.

$$\phi(r,\theta,\varphi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left(A_{\ell m} r^{\ell} + B_{\ell m} r^{-\ell-1} \right) Y_{\ell m}(\theta, \varphi)$$

This is a complete set of functions

Strategy for solution of Poisson eq

$$\nabla^2 \phi = -4\pi\rho$$

then $\nabla^2(\phi + \phi_h) = -4\pi\rho$

special solution

$$\vec{\nabla}^2 \phi_h = 0$$

Generally we have a charge distribution and boundary conditions

The ϕ_h are determined by the boundary conditions.

outside charge distribution ; $\rho = 0$

$\Rightarrow \vec{\nabla}^2 \phi = 0$ $\quad$ $\phi \to 0$ for $r \to \infty \Rightarrow A_{\ell m} = 0$

inside charge distribution : solution should be regular $\Rightarrow B_{\ell m} = 0$

Lecture # 5

$$\vec{\nabla}^2 \phi = 0$$

$$\vec{\nabla}^2 \phi = \frac{1}{r} \partial_r^2 (r\phi) + \frac{1}{r^2 \sin\theta} \partial_\theta \sin\theta \frac{\partial \phi}{\partial \theta} + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 \phi}{\partial \varphi^2}$$

$$\phi = \frac{u(r)}{r} P(\theta) Q(\varphi)$$

$$u(r) = \begin{cases} r^{\ell+1} \\ \frac{1}{r^\ell} \end{cases} \qquad Q(\varphi) = e^{im\varphi}$$

$$+ \sin\theta \, \partial_\theta (\sin\theta \, \partial_\theta P) + \ell(\ell+1) \sin^2\theta \, P - m^2 P = 0$$

$P = P(\cos\theta)$ is equation for Legendre polynomials

$$P = \cos\theta \Rightarrow \sin\theta \, \partial_\theta (\sin\theta \, \partial_\theta P) = -2\sin^2\theta \cos\theta$$

$\Rightarrow$ solution for $\ell = 1$ and $m = 0$.

Today: - Uniqueness
- Examples.

$r < a \qquad \phi = \sum A_{l0}\, r^l\, Y_{l0}(\theta, \varphi)$

$r > a \qquad \phi = \sum B_{l0} \frac{1}{r^{l+1}} Y_{l0}(\theta, \varphi)$

$$Y_{l0}(0, \varphi) = \sqrt{\frac{2l+1}{4\pi}}$$

$r < a$ on z axis $\quad \phi(z) = \frac{q}{a-z} = \sum_{n=0}^{\infty} \frac{q}{a}\left(\frac{z}{a}\right)^n$

$$\Rightarrow \quad A_{l0} = \frac{q}{a^{l+1}} \sqrt{\frac{4\pi}{2l+1}}$$

$r > a$ on z axis $\quad \phi_z = \frac{q}{z-a} = q \sum_{n=0}^{\infty} \frac{a^n}{z^{n+1}}$

$$\Rightarrow \quad B_{l0} = q\, a^l \sqrt{\frac{4\pi}{2l+1}}$$

potential in entire space is then given by

$$\phi(r<a) = \sum_{l=0}^{\infty} \frac{q\, r^l}{a^{l+1}} \sqrt{\frac{4\pi}{2l+1}}\, Y_{l0}(\theta, \varphi)$$

$$\phi(r>a) = \sum_{l=0}^{\infty} \frac{q\, a^l}{r^{l+1}} \sqrt{\frac{4\pi}{2l+1}}\, Y_{l0}(\theta, \varphi)$$

do not depend on φ

Some properties of spherical harmonics

$$Y_{\ell m}(\theta,\varphi) = \sqrt{\frac{2\ell+1}{4\pi}}\sqrt{\frac{(\ell-m)!}{(\ell+m)!}}\, P_{\ell m}(\cos\theta)\, e^{im\varphi}$$

$$r^2 \vec{\nabla}^2 Y_{\ell m}(\theta,\varphi) = -\ell(\ell+1) Y_{\ell m}(\theta,\varphi)$$

$$Y_{\ell -m}(\theta,\varphi) = (-1)^m Y^*_{\ell m}(\theta,\varphi)$$

orthogonality

$$\int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta\, d\theta\, Y^*_{\ell' m'}(\theta,\varphi)\, Y_{\ell m}(\theta,\varphi) = \delta_{\ell\ell'}\delta_{mm'}$$

completeness

$$\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Y^*_{\ell m}(\theta',\varphi')\, Y_{\ell m}(\theta,\varphi) = \delta(\varphi-\varphi') \times \delta(\cos\theta - \cos\theta')$$

⇒ An arbitrary function of θ and φ can be expressed as

$$f(\theta,\varphi) = \sum a_{\ell m} Y_{\ell m}(\theta,\varphi)$$

sum rule

$$\sum_{m=-\ell} Y_{\ell m}(\theta,\varphi)\, Y^*_{\ell m}(\theta,\varphi) = \frac{2\ell+1}{4\pi}$$

Example 1

charge q at $r=0$

$$\rho(r) = q\,\delta^3(r)$$

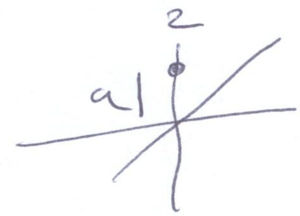
potential is spherically symmetric

$\Rightarrow$ only $\ell = 0$ and $m=0$ are allowed

$$\Rightarrow \phi = A_{00} + \frac{B_{00}}{r}$$

$\phi \to 0$ for $r \to \infty$ $\Rightarrow$ $A_{00} = 0$

Example 2 charge q at $z=a$



$$\phi = \sum \left(A_{\ell m} r^\ell + \frac{B_{\ell m}}{r^{\ell+1}} \right) Y_{\ell m}(\theta,\varphi)$$

axial symmetry $\Rightarrow$ $m=0$

We split the space in $r<a$ and $r>a$

$r<a$; potential is finite $\Rightarrow$ $B_{\ell m} = 0$

$r>a$; potential vanishes for $r\to\infty$

$\Rightarrow A_{\ell m} = 0$

We are going to determine the coëfficients from the potential on the z axis where it is given by $\phi(z) = \dfrac{q}{|z-a|}$

$r < a \qquad \phi = \sum A_{\ell 0}\, r^{\ell}\, Y_{\ell 0}(\theta, \varphi)$

$r > a \qquad \phi = \sum B_{\ell 0} \frac{1}{r^{\ell+1}} Y_{\ell 0}(\theta, \varphi)$

$$Y_{\ell 0}(0, \varphi) = \sqrt{\frac{2\ell+1}{4\pi}}$$

$r < a$ on z axis $\qquad \phi(z) = \frac{q}{a-z} = \sum_{n=0}^{\infty} \frac{q}{a}\left(\frac{z}{a}\right)^n$

$$\Rightarrow A_{\ell 0} = \frac{q}{a^{\ell+1}} \sqrt{\frac{4\pi}{2\ell+1}}$$

$r > a$ on z axis $\qquad \phi_z = \frac{q}{z-a} = q \sum_{n=0}^{\infty} \frac{a^n}{z^{n+1}}$

$$\Rightarrow B_{\ell 0} = q a^{\ell} \sqrt{\frac{4\pi}{2\ell+1}}$$

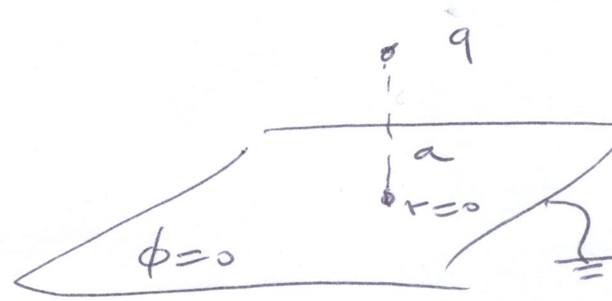
potential in entire space is then given by

$$\phi(r < a) = \sum_{\ell=0}^{\infty} \frac{q r^{\ell}}{a^{\ell+1}} \sqrt{\frac{4\pi}{2\ell+1}} Y_{\ell 0}(\theta, \varphi)$$

$$\phi(r > a) = \sum_{\ell=0}^{\infty} \frac{q a^{\ell}}{r^{\ell+1}} \sqrt{\frac{4\pi}{2\ell+1}} Y_{\ell 0}(\theta, \varphi)$$

do not depend on φ

Example 3



ϕ is axially symmetric $\Rightarrow m = 0$

again split space in $r<a$ and $r>a$

$r<a$ potential is finite $\Rightarrow B_{em} = 0$

$$\phi = \sum_{e=0}^{\infty} A_{e0} r^{e} Y_{e0}(\theta, \varphi)$$

$r>a$ $\quad \phi \to 0$ for $r \to \infty \Rightarrow A_{em} = 0$

$$\Rightarrow \phi = \sum_{e=0}^{\infty} B_{e0} \frac{1}{r^{e+1}} Y_{e0}(\theta, \varphi)$$

Potential vanishes on the plane

so $\phi(\theta = \frac{\pi}{2}) = 0$

$$Y_{e0}\left(\frac{\pi}{2}, \varphi\right) = (1 + (-1)^{e})(-1)^{e/2} c_e$$

$\Rightarrow$ e must be odd

$$\Rightarrow r<a \quad \phi = \sum_{e=0}^{\infty} A_{e0} (1 - (-1)^{e}) r^{e} Y_{e0}(\theta, \varphi)$$

on z axis $\quad z \to a$ then $\phi \to \frac{q}{a-z} = \sum_n \frac{q z^n}{a^{n+1}}$

$$\Rightarrow A_{e0} = \frac{q}{a^{e+1}} \sqrt{\frac{4\pi}{2e+1}} \qquad \left(Y_{e0}(\theta = 0, \varphi) = \sqrt{\frac{2e+1}{4\pi}}\right)$$

$$\Rightarrow r < a \quad \phi(\theta,\varphi) = \sum_{\ell=0}^{\infty} \frac{q}{a^{\ell+1}} r^{\ell} \sqrt{\frac{4\pi}{2\ell+1}} \left(1 - (-1)^{\ell}\right) Y_{\ell 0}(\theta,\varphi)$$

$r > a$ on z axis $\quad z \to a \quad \phi \to \frac{q}{z-a} = \sum_n \frac{q\, a^n}{z^{n+1}}$

$$\Rightarrow A_{\ell 0} = q\, a^{\ell} \sqrt{\frac{4\pi}{2\ell+1}}$$

$$\phi(\theta,\varphi) = \sum_{\ell=0}^{\infty} \frac{q\, a^{\ell}}{r^{\ell+1}} \sqrt{\frac{4\pi}{2\ell+1}} \left(1 - (-1)^{\ell}\right) Y_{\ell 0}(\theta,\varphi)$$

The first term is exactly the potential of a charge at $r = a$.

What about the $(-1)^{\ell}$ term

$$(-1)^{\ell}\, Y_{\ell 0}(\theta,\varphi) = Y_{\ell 0}(\pi-\theta,\varphi)$$

$$r < a \quad \phi(r) = \sum_{\ell=0}^{\infty} \frac{(-q)}{a^{\ell+1}} r^{\ell} \sqrt{\frac{4\pi}{2\ell+1}}\, Y_{\ell 0}(\pi-\theta,\varphi)$$

$$r > a \quad \phi(r) = \sum_{\ell=0}^{\infty} \frac{(-q)}{r^{\ell+1}} a^{\ell} \sqrt{\frac{4\pi}{2\ell+1}}\, Y_{\ell 0}(\pi-\theta,\varphi)$$

potential of charge $-q$ at $z = -a$

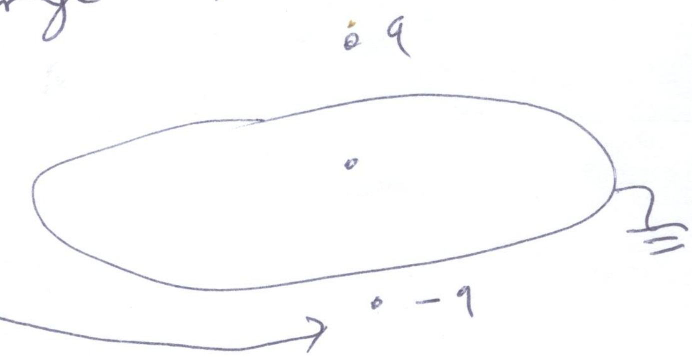
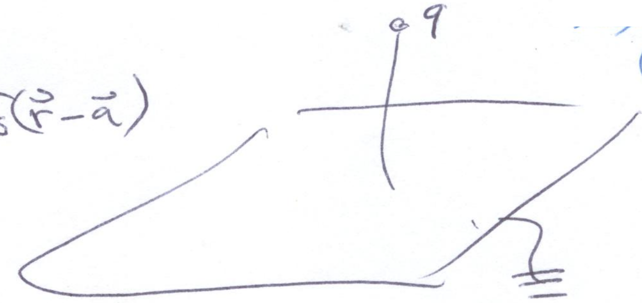


image charge

Alternate Method



$$\nabla^2(\phi_q + \phi_H) = -4\pi\,\delta(\vec{r}-\vec{a})$$

$$\phi_q = \frac{q}{|\vec{r}-\vec{a}|}$$

$$\vec{\nabla}^2 \phi_H = 0 \quad \Rightarrow \quad \phi_H = \sum_\ell \left(A_{\ell 0} r^\ell + \frac{B_{\ell 0}}{r^{\ell+1}}\right) Y_{\ell 0}$$

$$\phi_q \underset{r<a}{=} q \sum a^{-\ell-1} r^\ell \sqrt{\frac{4\pi}{2\ell+1}}\, Y_{\ell 0}$$

$$\underset{r>a}{=} q \sum r^{-\ell-1} a^\ell \sqrt{\frac{4\pi}{2\ell+1}}\, Y_{\ell 0}$$

$\phi_q + \phi_H = 0$ on $z=0$ plane ie for $\theta = \frac{\pi}{2}$

$$Y_{\ell 0}\left(\theta=\tfrac{\pi}{2}, \varphi\right) = \left(1+(-1)^\ell\right) c_\ell$$

vanishes for odd ℓ.

$\Rightarrow$ $A_{\ell 0}$ and $B_{\ell 0}$ for odd ℓ are not determined by the boundary conditions

$r<a$ $\quad A_{\ell} = -q \dfrac{1}{a^{\ell+1}} \sqrt{\dfrac{4\pi}{2\ell+1}}$
ℓ is even

on z axis $\quad \displaystyle\sum_{\ell=0}^{\infty} -q \frac{r^{2\ell}}{a^{2\ell+1}} \sqrt{\frac{4\pi}{4\ell+1}} \sqrt{\frac{4\ell+1}{4\pi}} = \frac{-q}{a\left(1-\frac{r^2}{a^2}\right)}$

$$= \frac{-qa}{a^2-r^2} = \frac{-qa}{2a}\left(\frac{1}{a-r} + \frac{1}{a+r}\right)$$

There can be no singularity at $r=a$

This has to be cancelled by the odd ℓ.

$$\sum_{\ell=0}^{\infty} A_{2\ell+1\,0}\, r^{2\ell+1} \sqrt{\frac{2(2\ell+1)+1}{4\pi}} \quad \Bigg| \quad A_{2\ell+1\,0} = \sqrt{\frac{4\pi}{2(2\ell+1)}}\, \frac{1}{a^{2\ell+2}}\, q$$

$$\sum_{\ell=0}^{\infty} \frac{q\, r^{2\ell+1}}{a^{2\ell+2}} = \frac{qr}{a^2} \frac{1}{1-\frac{r^2}{a^2}} = \frac{qr}{a^2-r^2}$$

combining even and odd givs

$$A_{\ell 0} = -q \frac{(-1)^\ell}{a^{\ell+1}} \sqrt{\frac{4\pi}{2\ell+1}}$$

on the z axis this give $\dfrac{q\,(r-a)}{a^2-r^2} = \dfrac{-q}{a+r}$

this is the potential of charge $-q$ at $z=-a$

$$\Rightarrow \quad \Phi^H_{r<a} = \sum_{\ell=0}^{\infty} (-q) \frac{(-1)^\ell}{a^{\ell+1}} r^\ell \sqrt{\frac{4\pi}{2\ell+1}}\, Y_{\ell 0}(\theta, \varphi)$$

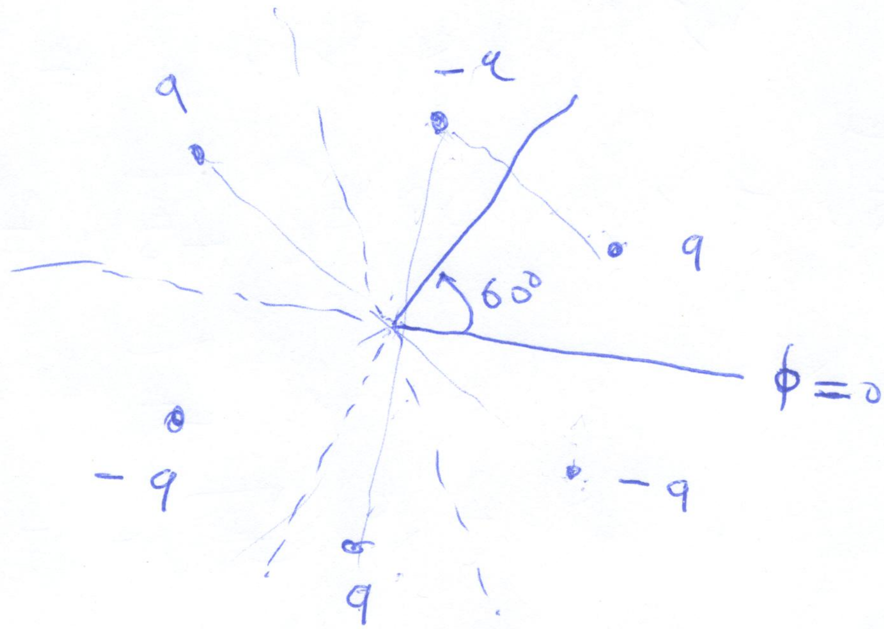
The analysis for $r>a$ is completely analogous.

This problem can be solved by image charges.
The trick is to find other charges that give the same boundary condition

$\Rightarrow$ Solution on the right of the plane is the same, not on the left

$$\phi(\vec{r}) = \frac{q}{|\vec{r}-\vec{a}|} - \frac{q}{|\vec{r}+\vec{a}|}$$

2nd Example of image charges

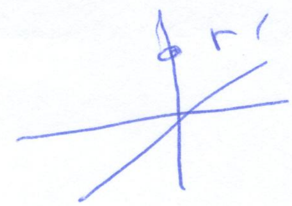


Expansion of $\frac{1}{|\vec{r}-\vec{r}'|}$ in spherical harmonics

choose $|\vec{r}|>|r'|$

$$\frac{1}{|\vec{r}-\vec{r}'|} = \sum \frac{1}{r^{\ell+1}} A_{\ell m} P_\ell Y_{\ell m}(\theta,\varphi)$$

choose $\vec{r}'$ on z axis



on z axis: $\frac{1}{|r-r'|} = \sum_{k=0} \frac{r'^k}{r^{k+1}}$

$\cancel{\Rightarrow}$ no dependence on $\varphi \Rightarrow m=0$

$$\Rightarrow A_{\ell 0} = \frac{r'^\ell}{Y_{\ell 0}(0,\varphi)} = r'^\ell \left(\frac{4\pi}{2\ell+1}\right)^{1/2} Y_{\ell 0}(0,\varphi)$$

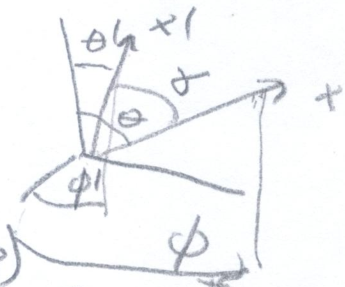
$$\underbrace{\frac{1}{|\vec{r}-\vec{r}'|}}_{\text{only depends on angle between } \vec{r} \text{ and } \vec{r}'} = \sum \frac{1}{r^{\ell+1}} r'^\ell \frac{4\pi}{2\ell+1} Y_{\ell 0}(0,\varphi)\, Y_{\ell 0}(\theta,\varphi)$$

$$Y_{\ell 0}(\theta',\varphi)\, Y_{\ell 0}(\theta,\varphi)$$

$Y_{\ell 0}(0,0) = \sqrt{\frac{2\ell+1}{4\pi}}$ $\qquad Y_{\ell 0}(\theta,\varphi) = \sqrt{\frac{2\ell+1}{4\pi}} P_\ell(\cos\theta)$

addition theorem

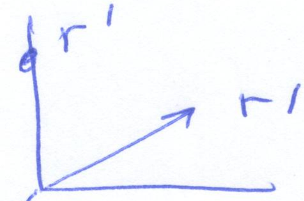
$$P_\ell(\cos\gamma) = \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{\ell} Y^*_{\ell m}(\theta',\varphi') \times Y_{\ell m}(\theta,\varphi)$$



$$\Rightarrow \frac{1}{|\vec{r}-\vec{r}'|} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{4\pi}{(2\ell+1)} \frac{r'^\ell}{r^{\ell+1}} Y^*_{\ell m}(\theta',\varphi') Y_{\ell m}(\theta,\varphi)$$

$$\frac{1}{|\vec{r}-\vec{r}'|} = \sum_{\ell} \frac{r'^{\ell}}{r^{\ell+1}} \sqrt{\frac{4\pi}{2\ell+1}}\, Y_{\ell 0}(\theta, \varphi)$$

now rotate $r' \to (\theta', \varphi')$



then $Y_{\ell m}(\vec{r}) = \sum_{m''} \mathcal{D}^{\ell}_{m'' m}\, Y_{\ell m''}\big(\underbrace{R_{\theta'\varphi'}(\theta,\varphi)}_{\text{new } \theta, \varphi \text{ coordinates of } \vec{r}}\big)$

$m \downarrow 0$, $\quad m \downarrow 0$

Wigner $\mathcal{D}$ matrices

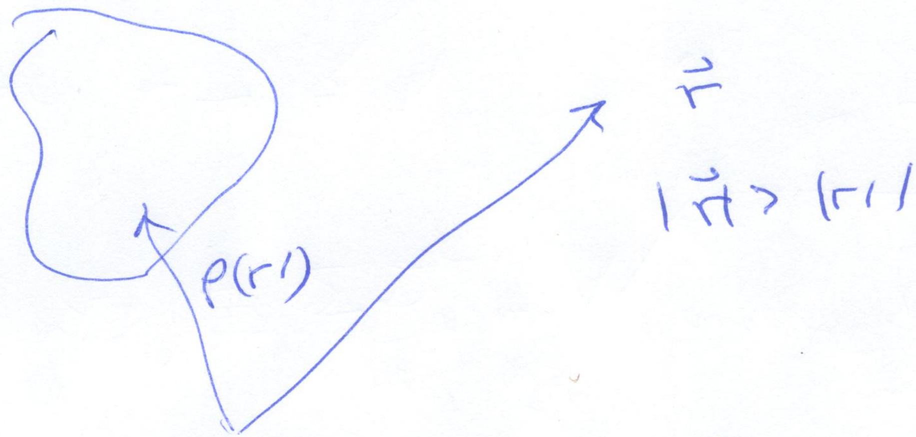
$$\mathcal{D}^{\ell}_{m'' 0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y^{*}_{\ell m''}(\theta', \varphi')$$

$$\Rightarrow \frac{1}{|\vec{r}-\vec{r}'|} = \sum_{\ell, m''} \frac{r'^{\ell}}{r^{\ell+1}} \frac{4\pi}{2\ell+1}\, Y^{*}_{\ell m''}(\theta', \varphi')\, Y_{\ell m''}(\theta, \varphi)$$

note that $r > r'$

Multipole expansion

a)



$$\phi(r') = \int d^3r' \, \rho(r') \frac{1}{|\vec{r}\,' - \vec{r}\,|}$$

$$= \int d^3r' \, \rho(r') \sum_{\ell m} \frac{r'^{\ell}}{r^{\ell+1}} \frac{4\pi}{2\ell+1} \rho(r') Y^*_{\ell m}(\theta', \varphi') \times Y_{\ell m}(\theta, \varphi)$$

multipole moment

$$Q_{\ell m} = \int d^3r' \, r'^{\ell} \rho(\vec{r}\,') \, Y^*_{\ell m}(\theta' \varphi')$$

$$\Rightarrow \phi(r) = \sum_{\ell=0}^{\infty} \frac{4\pi}{2\ell+1} \frac{1}{r^{\ell+1}} \sum_{m=-\ell}^{\ell} Y_{\ell m}(\theta, \varphi) \, Q_{\ell m}$$

$\ell = 0$ monopole

$\ell = 1$ dipole

$\ell = 2$ quadrapole

b) <u>Dipole moment</u>

$$\vec{p} = \int d^3x' \, \rho(x') \, \vec{x}'$$

$$Q_{10} = \sqrt{\frac{3}{4\pi}} \int z' \rho(x') \, d^3x' = \sqrt{\frac{3}{4\pi}} \, p_z$$

$$Q_{11} = -\sqrt{\frac{3}{8\pi}} \int (x' - iy') \, \rho(x') \, d^3x'$$

$$= -\sqrt{\frac{3}{8\pi}} \, (p_x - i p_y)$$

$$Q_{1-1} = - Q_{11}^{*} = \sqrt{\frac{3}{8\pi}} \, (p_x + i p_y)$$

The potential of a dipole is given by

$$\phi(r) = \underset{\substack{\uparrow \\ 2l+1}}{\frac{4\pi}{3}} \frac{1}{r^2} \Big(Y_{11} \left(-\sqrt{\frac{3}{8\pi}}\right) (p_x - i p_y) + Y_{10} \sqrt{\frac{3}{4\pi}} \, p_z + Y_{1-1} \sqrt{\frac{3}{8\pi}} (p_x + i p_y) \Big)$$

$$Y_{11} = -\sqrt{\frac{3}{8\pi}} \, (x + iy)/r \qquad Y_{10} = \sqrt{\frac{3}{4\pi}} \, \frac{z}{r}$$

$$Y_{1-1} = \sqrt{\frac{3}{8\pi}} \, \frac{(x - iy)}{r}$$

$$\phi(r) = \frac{4\pi}{3} \frac{1}{r^3} \Big((x+iy)(p_x - i p_y) \frac{3}{8\pi} + z \, p_z \frac{3}{4\pi} + (x - iy)(p_x + i p_y) \frac{3}{8\pi}$$

$$= \frac{1}{r^3} (x p_x + y p_y + z p_z) = \frac{1}{r^3} (\vec{r} \cdot \vec{p})$$