Lecture #21

(65a)

$$\vec{F} = \int d^3r\, \rho \vec{E} + \frac{1}{c}\vec{j}\times\vec{B}\ )$$

$$F_i = -\int_V \frac{d^3r}{4\pi c}\,\partial_t(\vec{E}\times\vec{B}) + \int_{\partial V} ds\, n_u T_{iu}$$

$$\vec{p} = \frac{\vec{E}\times\vec{B}}{4\pi c}$$

$$-\frac{1}{4\pi}\int d^3r\, \vec{E}\,\partial_t \vec{B} + \vec{H}\,\partial_t \vec{D} =$$

$$\int d^3r\, \vec{j}\cdot\vec{E} + \frac{c}{4\pi}\int d^3r\, \vec{\nabla}\cdot(\vec{E}\times\vec{H})$$

Today ME in vacuum
wave solutions

V

Time dependent ME in vacuum

$\rho = 0 \quad \vec{j} = 0 \Rightarrow \operatorname{div}\vec{B} = 0 \quad \operatorname{div}\vec{E} = 0$

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\partial_t \vec{B} \Rightarrow \vec{\nabla} \times \partial_t \vec{E} = -\frac{1}{c}\partial_t^2 \vec{B}$$

$$\vec{\nabla} \times \vec{B} = \frac{1}{c}\partial_t \vec{E} \Rightarrow \vec{\nabla} \times \partial_t \vec{B} = \frac{1}{c}\partial_t^2 \vec{E}$$

$$\Rightarrow c\,\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = -\frac{1}{c}\partial_t^2 \vec{B}$$

$$-c\,\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \frac{1}{c}\partial_t^2 E$$

$$(\vec{\nabla} \times (\vec{\nabla} \times B))_i = \varepsilon_{ijk}\varepsilon_{kpq}\,\partial_j \partial_p B_q$$

$$= (\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp})\,\partial_j \partial_p B_q$$

$$= \underbrace{\partial_j \partial_i B_j}_{\partial_i(\underbrace{\partial_j B_j}_{=0})} - \partial_j^2 B_i$$

$$\Rightarrow \vec{\nabla}^2 \vec{B} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{B} = 0$$

same $\vec{\nabla}^2 \vec{E} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{E} = 0$

These are wave equations.

Each component satisfies

$$(\partial_x^2 + \partial_y^2 + \partial_z^2) f - \frac{1}{c^2}\partial_t^2 f = 0$$

Normally we expand in plane waves

$$\sum_k a_k \, e^{i(\vec{k}\cdot\vec{x} - \omega t)}$$

Since the equations are linear we can solve them component by component

$$\Rightarrow \quad \vec{E}_k = \mathrm{Re}\, \vec{e}_k \, e^{i(\vec{k}\vec{x} \pm \omega t)}$$

polarization of the wave

$$\vec{\nabla} \times \vec{B} = \frac{1}{c}\partial_t \vec{E}$$

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\partial_t \vec{B}$$

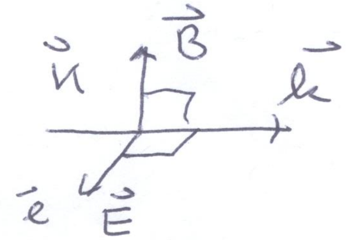
$$(i\vec{k} \times \vec{e}) \, e^{i(\vec{k}\vec{x} - \omega t)} = -\frac{1}{c}\partial_t \vec{B}$$

$$\Rightarrow \quad B = (i\vec{k} \times \vec{e}) \, e^{i\vec{k}\vec{x}} \, \frac{c}{i\omega} \, e^{-i\omega t}$$

physical component is real,

$$\Rightarrow B = \mathrm{Re}\left(\frac{c\vec{k}}{\omega} \times \vec{e}\right) e^{i\vec{k}\vec{x} - i\omega t}$$

wave



propagation direction

$$\frac{1}{c}\partial_t \vec{E} = \vec{\nabla} \times \vec{B} = \frac{i\vec{k}c}{i\omega} \times (c\vec{k} \times \vec{e}) \, e^{i\vec{k}\vec{x} - i\omega t}$$

why is $\vec{E} \perp \vec{k}$?

$\vec{\nabla} \cdot \vec{E} = 0 \Rightarrow \vec{k} \cdot \vec{e} \, e^{i(kx - \omega t)} = 0$

$\Rightarrow \vec{k} \cdot \vec{e} = 0$

Same for $\vec{B}$, $\vec{\nabla} \cdot \vec{B} = 0 \Rightarrow \vec{B} \perp \vec{k}$

<u>polarization</u> direction of $\vec{e}$

$\vec{e} \perp \vec{k} \Rightarrow \exists$ two independent directions

- elliptical polarization $\vec{e} = \vec{e}_1 + i \vec{e}_2$
- linear polarization $\vec{e}_1 \parallel \vec{e}_2$
- circular polarization $\vec{e}_1 \perp \vec{e}_2$

$\vec{e}_R = \vec{e}_x + i \vec{e}_y$ right handed polarical

$\vec{e}_L = \vec{e}_x - i \vec{e}_y$ left handed polarized.

$\mathrm{Re}\, \vec{E} = \mathrm{Re}\,(\vec{e}_x + i\vec{e}_y)\, e^{-i\omega t}$

$= \vec{e}_x \cos\omega t + \vec{e}_y \sin\omega t$

this is the equation of a circle

Lecture #22

ME in vacuum $\rho = 0$ $\vec{J} = 0$

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\partial_t \vec{B} \qquad \mathrm{div}\,\vec{E} = 0$$

$$\vec{\nabla} \times \vec{B} = \frac{1}{c}\partial_t \vec{E} \qquad \mathrm{div}\,\vec{B} = 0$$

$$\Rightarrow \vec{\nabla}^2 \vec{E} - \frac{1}{c^2}\partial_t^2 \vec{E} = 0$$

$$\vec{\nabla}^2 \vec{B} - \frac{1}{c^2}\partial_t^2 \vec{B} = 0$$

plane waves $\vec{E} = \vec{e}\, e^{i\vec{k}\cdot\vec{x} - i\omega t}$

$\vec{B} \perp E$ $\quad (\vec{k} \times \vec{E} \parallel B$

$\vec{\nabla} \cdot \vec{E} = 0 \Rightarrow E = \perp \vec{k}$

$\vec{e}$ is polarization

Today: time dependent ME in matter

Time dependent Maxwell equations in matter in the dipole approximation

approximation matter can be represented by a dipole density and a magnetic moment density

The dipole density gives $\rho_{bound} = -\vec{\nabla}_r \vec{P}$

and the magnetic moment density
$$j_b = c\, \vec{\nabla} \times \vec{M}$$

$$\Rightarrow 1)\ \vec{\nabla} \times \vec{E} = -\frac{1}{c}\partial_t \vec{B}$$
$$2)\ \vec{\nabla} \cdot \vec{B} = 0$$
$$3)\ \vec{\nabla} \cdot \vec{E} = 4\pi(\rho - \vec{\nabla}\vec{P})$$
$$4)\ \vec{\nabla} \times \vec{B} = \frac{4\pi}{c}\vec{j} + 4\pi \vec{\nabla} \times \vec{M}$$

(no need to include surface charge density and surface current because they can be represented as δ-functions.

However as in case of the vacuum eqs, they are not consistent

1) and 2) are consistent
$$0 = \vec{\nabla} \cdot \vec{\nabla} \times \vec{E} = -\frac{1}{c}\partial_t \vec{\nabla}\vec{B} = 0$$

but
$$0 = \vec{\nabla} \cdot \vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \underbrace{\vec{\nabla} \vec{j}}_{= -\frac{\partial \rho}{\partial t}} + \underbrace{\frac{4\pi}{c}\vec{\nabla} j_b}_{\text{add additional term}} + \overbrace{\vec{\nabla} c\, \vec{\nabla} \times \vec{M}}^{=0}$$

From 3) $\frac{\partial \rho}{\partial t} = \frac{1}{4\pi}\partial_t \vec{\nabla}\vec{E} + \frac{1}{4\pi}\partial_t \nabla\vec{P}$

So we need

$$-\frac{4\pi}{c}\frac{1}{4\pi}\vec{\nabla}(\partial_t\vec{E}+\partial_t\vec{P}) + \frac{4\pi}{c}\vec{\nabla}\vec{j}_D = 0$$

$$\Rightarrow \vec{j}_D = \frac{1}{4\pi}(\partial_t\vec{E}+\partial_t\vec{P}) + \vec{\nabla}\times\vec{Q}$$

choose $Q = 0$

$\Rightarrow$

$$\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\partial_t\vec{E} + \frac{1}{c}\partial_t\vec{P} + 4\pi\vec{\nabla}\times M$$

$\Rightarrow$

$$\vec{\nabla}\times(\overbrace{\vec{B}-4\pi\vec{M}}^{\vec{H}}) = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\partial_t\vec{D}$$

$$\vec{\nabla}\cdot(\underbrace{\vec{E}+\vec{P}}_{D}) = 4\pi\rho$$

We have rewritten the ME in matter in terms of $\vec{D}$ and H:

$$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\partial_t\vec{B}$$

$$\vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\cdot\vec{D} = 4\pi\rho$$

$$\vec{\nabla}\times\vec{H} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\partial_t\vec{D}$$

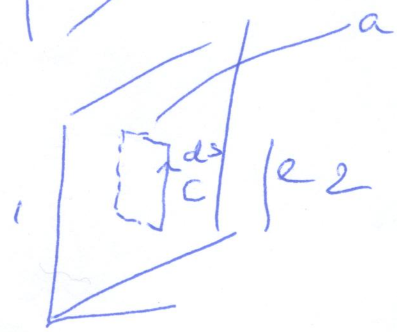
$\vec{P}$ is time dependent $\Rightarrow$ ε is time dependent

Boundary conditions

$$\vec{\nabla}\cdot\vec{B} = 0 \quad \Rightarrow \quad \Delta B_n = 0$$

$$\vec{\nabla}\cdot\vec{D} = 4\pi\rho \quad \Rightarrow \quad \Delta D_n = 4\pi\sigma$$

this can be proved using a Gaussian pillbox.



$$\vec{\nabla}\times\vec{E} = \partial_t \vec{B}$$

$$\oint_c \vec{E}\, d\vec{s} = \int_S \vec{\nabla}\times\vec{E}\cdot\vec{n}\, d\vec{a}$$

$$= \int_S \partial_t \vec{B}\, \vec{n}\, d\vec{a}$$

$$= \partial_t \int_S \vec{B}\cdot\vec{n}\, da$$

continuous

$$\oint \vec{E}\cdot d\vec{s} = l(E_{2\tan} - E_{1\tan}) = \partial_t B_n\, a\, l$$

$$a \to 0$$

$$\Rightarrow E_{2\tan} = E_{1\tan}.$$

For the $\vec{\nabla}\times\vec{H}$ we obtain

$$\int_S \vec{\nabla}\times\vec{H}\, d\vec{s} = \frac{4\pi}{c}\int_S j\cdot d\vec{s} + \underbrace{\frac{1}{c}\int \partial_t \vec{D}\cdot ds}_{O(a\,l)}$$

$$\oint_c \vec{H}\cdot ds \qquad \frac{4\pi}{c} I$$

$$(H_{2\tan} - H_{1\tan})\, l = \frac{4\pi}{c} K\, l \qquad a \to 0$$

$$\Rightarrow \Delta H_{tan} = \frac{4\pi}{c} K$$

Energy and momentum in matter

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\partial_t \vec{B}$$

$$\vec{\nabla} \times \vec{H} = \frac{1}{c}\partial_t \vec{D} + \frac{4\pi}{c}\vec{J}$$

note that the equations in matter are an approximation contrary to the equations in vacuum.

then

$$\frac{1}{c}\vec{E}\partial_t \vec{D} + \frac{1}{c}\vec{H}\partial_t \vec{B} = -\frac{4\pi}{c}\vec{J}\cdot\vec{E} + \vec{E}\cdot\vec{\nabla}\times\vec{H} - \vec{H}\cdot\vec{\nabla}\times\vec{E}$$

$$= -\frac{4\pi}{c}\vec{J}\cdot\vec{E} - \vec{\nabla}\cdot(\vec{E}\times\vec{H})$$

$$\vec{\nabla}\cdot\vec{E}\times\vec{H} = \partial_i \varepsilon_{ijk} E_j H_k$$

$$= \varepsilon_{ijk}(\partial_i E_j) H_k + \varepsilon_{ijk} E_j \partial_i H_k$$

$$= H\cdot\vec{\nabla}\times\vec{E} - \vec{E}\cdot\vec{\nabla}\times\vec{H}$$

integrate and $-\frac{c}{4\pi}$

$$-\frac{1}{4\pi}\int d^3r \left(\vec{E}\cdot\frac{\partial\vec{D}}{\partial t} + \vec{H}\cdot\frac{\partial\vec{B}}{\partial t}\right) = \int d^3r\, \vec{J}\cdot\vec{E} + \frac{c}{4\pi}\int d^3r\, \vec{\nabla}\cdot\vec{E}\times\vec{H}$$

assum liner media (72)

$$\vec{D} = \varepsilon \vec{E} \qquad \vec{B} = \mu \vec{H}$$

$$\Rightarrow \frac{\partial}{\partial t}\left(-\frac{1}{8\pi}\int d^3r\,(\vec{E}\cdot\vec{D} + \vec{B}\cdot\vec{H})\right)$$

extra factor $\frac{1}{2}$

$$= \int d^3r\,\vec{J}\cdot\vec{E} + \frac{c}{4\pi}\underbrace{\int d^3r\,\vec{\nabla}\cdot\vec{E}\times\vec{H}}_{\frac{c}{4\pi}\int_{\partial V} ds\,\hat{n}\cdot\vec{E}\times\vec{H}}$$

decrease of energy of the medium

power transferred to current

energy flow out of the volume

Poynting vector $\vec{P} = \frac{c}{4\pi}\vec{E}\times\vec{H}$

energy flow per unit area per second

Free Fields in isotropic media

We want to derive a wave equation w/o.

$$\varepsilon_{ij}(\omega) = \delta_{ij}\,\varepsilon(\omega)$$
$$\mu_{ij}(\omega) = \delta_{ij}\,\mu(\omega)$$
$$\sigma_{ij}(\omega) = \delta_{ij}\,\sigma(\omega)$$
$$\vec{j} = \sigma \vec{E}$$
$$\vec{D} = \varepsilon \vec{E}$$
$$B = \mu H$$

We consider monochromatic waves

$$\vec{E}(\vec{r},t) = \vec{E}(\vec{r})\,e^{i\omega t}$$
$$\vec{B}(\vec{r},t) = \vec{B}(\vec{r})\,e^{i\omega t}$$

$$\vec{E}(\vec{r},t) = \frac{1}{\sqrt{2\pi}}\int e^{-i\omega t}\,\vec{E}(\omega)\,d\omega$$

$$\partial_t \vec{E}(\vec{r},t) = \frac{1}{\sqrt{2\pi}}\int (-i\omega)\,e^{-i\omega t}\,\vec{E}(\omega)\,d\omega.$$

$\vec{E}(r,t)$ is real

$$\Rightarrow E^*(r,t) = E(r,t)$$
$$= \int e^{i\omega t}\,\vec{E}^*(\omega)\,d\omega = \int e^{-i\omega t}\,E(\omega)\,d\omega$$
$$\int e^{-i\omega t}\,\vec{E}^*(-\omega)\,d\omega$$
$$\Rightarrow E^*(-\omega) = E(\omega)$$

For a monochromatic wave we have

$$E = \left(e^{-i\omega t}E(\omega) + e^{i\omega t}E^*(-\omega)\right)\frac{1}{\sqrt{2\pi}} =$$
$$= \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\,\mathrm{Re}\,E(\omega)\,e^{-i\omega t}$$

Maxwell eq.

$$\vec{\nabla} \times \vec{H} = \frac{1}{c}\underbrace{\partial_t}_{-i\omega} \vec{D} + \frac{4\pi}{c}\vec{j} \qquad \Rightarrow \vec{\nabla}\vec{D} = 4\pi\rho$$

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\underbrace{\partial_t}_{-i\omega} B \qquad \Rightarrow \vec{\nabla}.\vec{B} = 0$$

material eqs $\vec{j} = \sigma \vec{E}$

$$\vec{D} = \varepsilon \vec{E}$$
$$\vec{B} = \mu H$$

$$\nabla \times \frac{\vec{B}}{\mu} = \frac{1}{c}(-i\omega\varepsilon + 4\pi\sigma)\vec{E}$$
$$= -\frac{i\omega}{c}\underbrace{\left(\varepsilon + \frac{4\pi\sigma i}{\omega}\right)}_{\equiv \varepsilon(\omega)}\vec{E}$$

Conductivity is described by the imaginary part of $\varepsilon(\omega)$

$$\Rightarrow \vec{\nabla} \times \frac{\vec{B}}{\mu} = -\frac{i\omega}{c}\varepsilon \vec{E}$$

$$\vec{\nabla} \times \vec{E} = \frac{i\omega}{c}\vec{B}$$

$$\Rightarrow \vec{\nabla} \times (\vec{\nabla} \times E) = \frac{i\omega}{c}\mu - \frac{i\omega}{c}\varepsilon\vec{E}$$
$$= \mu\varepsilon\frac{\omega^2}{c^2}\vec{E}$$

$$\vec{\nabla} \times (\vec{\nabla} \times B) = \mu\varepsilon\left(\frac{-i\omega}{c}\right)\frac{i\omega}{c}\vec{B} = \mu\varepsilon\frac{\omega^2}{c^2}\vec{B}$$

If there are no free charges we thus have the ME $\vec\nabla \cdot \vec E = 0$

$$-\vec\nabla^2 \vec E = \frac{\omega^2}{c^2}\mu\epsilon\vec E$$

$$\vec\nabla \cdot \vec E = 0$$

$$-\vec\nabla^2 B = \frac{\omega^2}{c^2}\mu\epsilon\vec B$$

$$\vec\nabla \cdot \vec B = 0$$

$$\vec B = \frac{c}{i\omega}\vec\nabla \times \vec E$$

$$\vec E = -\frac{c}{i\omega}\frac{\vec\nabla \times \vec B}{\epsilon\mu}$$

Because the equations are linear we can consider plane wave solutions

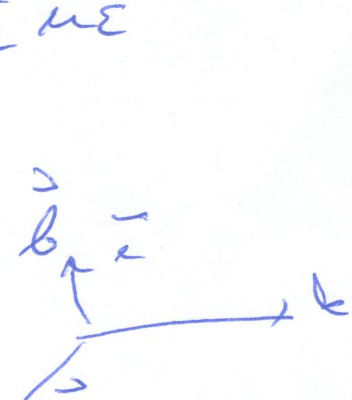
$$\vec E = e\, e^{i\vec k \vec x} \qquad \vec B = b\, e^{i\vec k \vec x}.$$

The general solution is given by a linear combination of these solution.

$$\vec\nabla^2 \to -k^2 \quad \Rightarrow \quad k^2 = \frac{\omega^2}{c^2}\mu\epsilon$$

$$\vec b = \frac{c}{i\omega}\, i\vec k \times \vec e$$

$$\vec e = -\frac{c}{\omega\epsilon\mu}\vec k \times \vec b$$



Energy density

$$\vec{E}_r = \mathrm{Re}\, \vec{e}_\lambda\, e^{i\vec{k}\cdot\vec{x} - i\omega t} = \mathrm{Re}\, \vec{E}$$

$$\vec{B}_r = \mathrm{Re}\, \frac{c\vec{k}}{\omega} \times \vec{e}_\lambda\, e^{i\vec{k}\vec{x} - i\omega t} = \mathrm{Re}\, B$$

energy density $\quad u = \varepsilon \dfrac{E_r^2}{8\pi} + \dfrac{B_r^2}{8\pi\mu}$

$$= \varepsilon \frac{(\vec{E} + \vec{E}^*)^2}{4\cdot 8\pi} + \frac{(\vec{B} + \vec{B}^*)^2}{4\cdot 8\pi}$$

$$\overline{E_r^2} \sim \overline{e^{2i\omega t}} \Rightarrow \overline{E_r^2} = 0 \quad \overline{E_r^{*2}} = 0$$

$$\overline{B_r^2} = 0 \quad \overline{B_r^{*2}} = 0$$

$$= u = \frac{\varepsilon}{16\pi} \vec{E}\cdot\vec{E}^* + \frac{\vec{B}\,\vec{B}}{16\pi\mu}$$

$$= \left(\frac{\varepsilon}{16\pi} + \frac{c^2 k^2}{\omega^2}\frac{1}{16\pi\mu}\right)\vec{E}\vec{E}^* = \frac{\varepsilon}{8\pi}\vec{e}_\lambda\cdot\vec{e}_\lambda^*$$

$$k^2 = \frac{\omega^2}{c^2}\mu\varepsilon$$

Energy flow $\quad \vec{P} = \dfrac{c}{4\pi}\vec{E}_r \times \vec{H}_r$

$$= \frac{c}{4\pi}\left(\frac{\vec{E} + \vec{E}^*}{2}\right) \times \frac{(\vec{H} + \vec{H}^*)}{2}$$

$$\overline{e^{\pm 2i\omega t}} = 0 \Rightarrow \vec{P} = \frac{c}{16\pi}\left(\vec{E}\times\vec{H}^* + \vec{E}^*\times\vec{H}\right)$$

$$\vec{E}\times\vec{H}^* = \vec{E}\times\frac{\vec{B}^*}{\mu} = \vec{E}\times\left(\frac{\vec{k}c}{\omega\mu}\times\vec{E}^*\right)$$

$$\vec{E}^*\times\vec{H} = \vec{E}^*\times\left(\frac{\vec{k}c}{\omega\mu}\times\vec{E}\right)$$

$$\vec{E} \times (\vec{k} \times \vec{E}^*) = \vec{e}\, e^{i\vec{k}\cdot\vec{x}} \times (\vec{k} \times \vec{e}^*)\, e^{-i\vec{k}\cdot\vec{x}}$$

$$= \vec{e} \times (\vec{k} \times \vec{e}^*)$$

$$= (\vec{e}\cdot\vec{e}^*)\,\vec{k} - \underbrace{(\vec{e}\cdot\vec{k})}_{=0}\,\vec{e}^*$$

$$\vec{E}^* \times (\vec{k} \times \vec{E}) = \vec{e}^* \times (\vec{k} \times \vec{e})$$

$$= (\vec{e}^*\cdot\vec{e})\,\vec{k} - \underbrace{(\vec{e}^*\cdot\vec{k})}_{=0}\,\vec{e}$$

$$\Rightarrow \vec{P} = \frac{c}{8\pi}\left(\vec{E}^* \times \vec{H}\right)$$

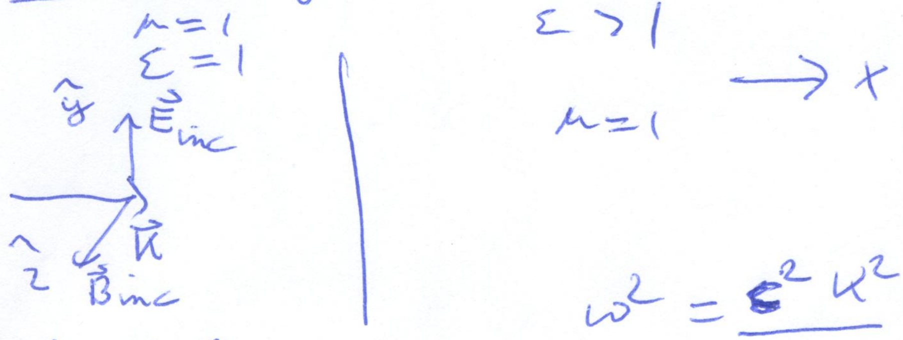
$$= \frac{c}{8\pi}\left(e_\lambda^* \times \left(\frac{c\vec{k}}{\omega} \times e_\lambda\right)\frac{1}{\mu}\right)$$

$$= \frac{\varepsilon}{8\pi}\frac{c^2\vec{k}}{\varepsilon\mu\omega}\,\vec{e}_\lambda^*\cdot\vec{e}_\lambda = \bar{u}\,\frac{c}{\sqrt{\varepsilon\mu}}\,\hat{k}$$

note that $\bar{u} = \frac{\varepsilon}{8\pi}\,\vec{e}_\lambda\cdot\vec{e}_\lambda^*$

$$k = \frac{\omega}{c}\sqrt{\mu\varepsilon}$$

Solution of ME for infinite slab



$$\omega^2 = c^2 k^2 \qquad\qquad \omega^2 = \frac{c^2 k^2}{\varepsilon \mu}$$

no free charges or currents

$\Rightarrow$ $\vec{E}_{tan}$ is continuous
$\vec{B}_{tan}$ " "
$\vec{E}_n$ is continuous
$\vec{B}_n$ is continuous

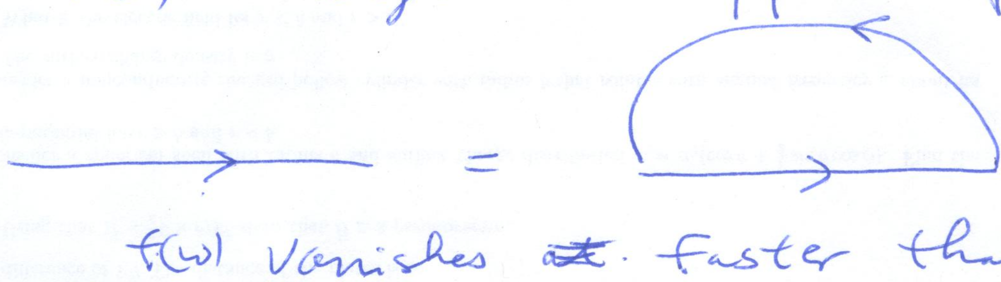
$\Big\}$ $\Rightarrow$ $\vec{E}$ and $\vec{B}$ are continuous at $\vec{x} = 0$

Initial condition: only nonzero $\vec{E}$ field for $x < 0$ and $t = 0$

$$\vec{E}_{inc} = \hat{y} \int_{-\infty}^{\infty} d\omega \, f(\omega) \, e^{i\left(\frac{\omega}{c}\right)(x - ct)}$$

chose $f(\omega)$ analytic in upper half plane

ω



$f(\omega)$ vanishes ~~at~~ faster that $\frac{1}{\omega}$ for $\omega \to \infty$. Then we can close the integration contour.
When $f(\omega)$ is analytic for $\mathrm{Im}(\omega) > 0$ the integral vanishes

For $x > 0$ we only have a right moving wave

$$\vec{E} = \hat{y} \int d\omega \, A'(\omega) \, e^{\frac{i\omega}{c}(x-ct)}$$

$\vec{E}$ is continuous ~~of~~ at $x = 0$

$$\Rightarrow \quad A'(\omega) = A(\omega) + B(\omega)$$

Definition of the reflection coefficient

$$-f(\omega) R(\omega) = B(\omega)$$

Transmission coefficient

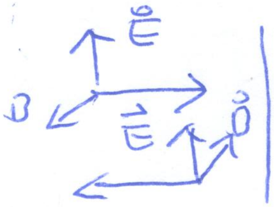
$$T(\omega) f(\omega) = A'(\omega)$$

$$\Rightarrow \quad f(\omega) \cancel{\#} T(\omega) = f(\omega) + -f(\omega) R(\omega)$$

$$\Rightarrow \quad R(\omega) + T(\omega) = 1$$

Next we look at the $\vec{B}$ field

$$\vec{B} = \frac{c\vec{k}}{\omega} \times \vec{E}$$



For the reflected wave,
the signe of $\hat{k}$ is opposite $\Rightarrow$
also $\vec{B}$ points in the opposite direction.
whil $\vec{E}$ of the reflected wave remains
in the same direction

$\vec{E}_{inc} = 0$ for $x > ct$, so causality is satisfied

$\vec{B}_{inc} = \hat{z}\, \vec{E}_{inc}$.

ME $(\vec{\nabla} \times \vec{E}) = -\frac{1}{c}\partial_t B$

only z component nonzero $\partial_x E_y = -\frac{1}{c}\partial_t B_z$

$\vec{\nabla} \times \vec{B} = \frac{1}{c}\partial_t\, D$.

only ~~y~~ component is nonzero

$\partial_x B_z = -\frac{1}{c}\partial_t D_y$

$\Rightarrow \partial_x^2 E_y = -\frac{1}{c}\partial_t \partial_x B_z = \frac{1}{c^2}\partial_t^2 D_y$

general solution for $x < 0$

$$\vec{E} = \hat{y}\int d\omega \left(A(\omega)\, e^{\frac{i\omega}{c}(x-ct)} + e^{\frac{i\omega}{c}(-x-ct)} B(\omega) \right)$$

right moving wave left moving wave

For $t < 0$ we do not have a left moving wave and also for $x > 0$ we do not have a left moving wave

This is achieved when $B(\omega)$ is analytic in the upper half plane of ω.

When x is more negative than ct, there is also no left moving wave as dictated by causality

So we obtain the $\vec{B}$ field

$$x<0: \vec{B} = \frac{c\vec{k}}{\omega} \times \hat{y} \int d\omega\, f(\omega)\, e^{\frac{i\omega}{c}(x-ct)} - B(\omega)\, e^{-\frac{i\omega}{c}(x+ct)}$$

has opposite sign.

$$x>0: \quad \vec{B} = \frac{c\vec{k}'}{\omega} \times \hat{y} \int d\omega\, A'(\omega)\, e^{ik'x - i\omega t}$$

$\vec{B}$ is continuous at $x=0$.

$$\Rightarrow \quad f(\omega) - B(\omega) = \frac{k'}{k} A'(\omega)$$

$$\Rightarrow f(\omega)\big(1+R(\omega)\big) = \sqrt{\varepsilon}\, T(\omega)\, f(\omega)$$

$$\Rightarrow 1+R(\omega) = \sqrt{\varepsilon}\, T(\omega)$$

we already had $R(\omega) + T(\omega) = 1$

$$\Rightarrow T(\omega) = \frac{2}{1+\sqrt{\varepsilon}}$$

$$R(\omega) = \frac{\sqrt{\varepsilon}-1}{\sqrt{\varepsilon}+1}$$