model

Magnetic fields in matter $\vec{M}(\vec{r})$

$$\stackrel{\wedge}{=} c\vec{\nabla}_r \times \vec{M} \qquad + c\,\vec{M} \times \hat{n}$$

volume current

surface

$$\vec{B} \to \vec{B} - 4\pi\vec{M} \equiv \vec{H}$$

permanent magnets
paramagnetism
diamagnetism

Today Electromotive force

Electromotive Force

magnetic flux $\quad \Phi = \int_S \vec{B} \cdot d\vec{s}$

Flux is independent of the surface

prove

$$\int_V \operatorname{div} \vec{B} \, d^3r = 0 \quad \Rightarrow \int_{\partial V} \vec{B} \cdot d\vec{s} = 0$$

$$\Rightarrow \int_{S_1} \vec{B} \cdot d\vec{s} + \int_{S_2} \vec{B} \cdot d\vec{s} = 0$$

if we choose the normal in the same direction this gives

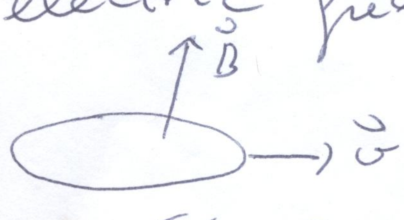
$$\int_{S_1} \vec{B} \cdot d\vec{s} - \int_{S_2} \vec{B} \, d\vec{s} = 0$$

qed

An electromotive force occurs if a conductor moves ~~through~~ through a magnetic field. The reason is that the electrons feel the Lorentz force

$$F_L = e \frac{\vec{v}}{c} \times \vec{B}$$

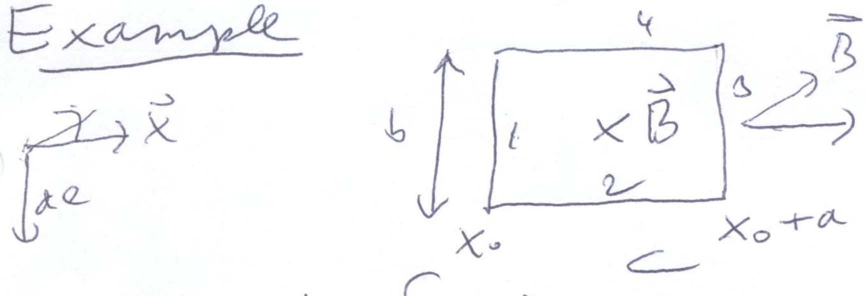
we can also interpret this as an effective electric field $\quad \vec{E}_{eff} = \frac{\vec{v}}{c} \times \vec{B}$



Electromotive force

$$\mathcal{E}_{ef} = \oint_C \vec{E} \cdot d\vec{\ell} = \int \frac{\vec{v}}{c} \times \vec{B} \cdot d\vec{\ell}$$

Example



$\vec{B} = \text{constant}$

$$\frac{1}{c}\oint_C \vec{v} \times \vec{B} \cdot d\vec{\ell} = \frac{1}{c}\frac{d}{dt}\oint_C \vec{x} \times \vec{B} \cdot d\vec{\ell}$$

$$= \frac{1}{c}\frac{d}{dt}\oint_C \vec{B} \cdot d\vec{\ell} \times \vec{x}$$

integral only nonzero on vertical sides

$$= \frac{1}{c}\frac{d}{dt}\Big(\underbrace{\int_1 \vec{B} \cdot d\vec{\ell} \times \vec{x}}_{B b x_0} + \underbrace{\int_3 \vec{B} \cdot d\vec{\ell} \times \vec{x}}_{- B b (x_0 + a)}$$

$$= -\frac{1}{c}\frac{d}{dt} B A = -\frac{1}{c}\frac{d}{dt}\underbrace{\Phi}_{\text{flux}}$$

Next we will prove this in general

$$\mathcal{E}_{ef} = -\frac{1}{c}\frac{d\Phi}{dt}$$

the flux changes due to a moving circuit.

<u>Faraday law</u> same formula but then the changes due to the time dependence of the magnetic field.

Lecture # 17

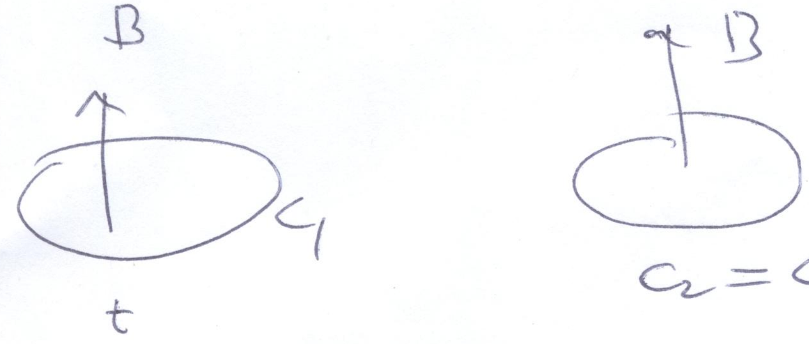
electromotive force

$$\mathcal{E} = -\frac{1}{c}\frac{d\Phi}{dt} \qquad \Phi = \int \vec{B}\cdot d\vec{A}$$

$$= \oint_c \vec{E}\cdot dl = \int \frac{\vec{v}}{c}\times\vec{B}\cdot d\vec{l}$$

Today: Faraday for arbitrary circuit

Flux $\phi = \int \underset{\vec{\nabla}\times\vec{A}}{\vec{B}} \cdot d\vec{s} = \int \vec{A} \cdot d\vec{\ell}$



$c_2 = c_1 + \delta c_1$

$c_1: \quad \vec{x}(\tau) \qquad\qquad x(0) = x(1)$

$c_2: \quad \vec{x}(\tau) + \delta x(\tau) \qquad 0 \le \tau < 1$

$d\vec{\ell} = \frac{d\vec{x}}{d\tau} d\tau$

$$\delta\phi = \oint_{c_2} \vec{A} \cdot d\vec{\ell} - \oint_{c_1} \vec{A} \cdot d\vec{\ell}$$

$$= \int_0^1 d\tau\, A_i(x(\tau) + \delta x(\tau)) \frac{d(\vec{x} + \delta x)_i}{d\tau}$$

$$- \int_0^1 d\tau\, A_i(x(\tau)) \cdot \frac{d x_i(\tau)}{d\tau}$$

$$= \int_0^1 d\tau \left(\partial_k A_i(\vec{x})\, \delta x_k \frac{dx_i}{d\tau} + A_i \frac{d(\delta x)_i}{d\tau} \right)$$

// partial integration

$$\left(\frac{-d}{d\tau} A_i\right)(\delta x_i)$$

$$= -\frac{dA_i}{dx_p} \frac{dx_p}{d\tau} \delta x_i$$

$$\Rightarrow \delta\phi = \int_0^1 d\tau \left(\partial_k A_i - \partial_i A_k \right) \times \delta x_k \frac{dx_i}{d\tau}$$

$$d\tau \frac{d\vec{x}}{d\tau} = \vec{d\ell}$$

$$\Rightarrow \delta\phi = \oint (\vec{d\ell})_i \, (\partial_k A_i - \partial_i A_k) \cdot \delta x_k$$

Let us look at $(\delta x \times \vec{B})_i = (\delta x \times (\vec{\nabla}\times\vec{A}))_i$

$$= \varepsilon_{ijk}\, \delta x_j \, \varepsilon_{kpq}\, \partial_{x_p} A_q$$

$$= (\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp})\, \delta x_j \, \partial_{x_p} A_q$$

$$= \delta x_q \, \partial_{x_i} A_q - \delta x_j \, \partial_{x_j} A_i$$

So $$\delta\phi = -\oint (\vec{\delta x} \times \vec{B}) \cdot d\vec{\ell}$$

$$\Rightarrow -\frac{1}{c}\frac{\delta\phi}{\delta t} = +\frac{1}{c}\oint \left(\frac{\vec{\delta x}}{dt} \times \vec{B}\right) \cdot \vec{d\ell}$$

$$= \frac{1}{c}\oint \frac{\vec{v}}{c} \times \vec{B} \cdot d\ell$$

$$= \mathcal{E}_f.$$

Faraday's law

move the magnet and Keep the circuit Fixed

Galileon invariance: Physics in frame A) and in frame B) is the same

$$Q = \int I\,dt = \int \frac{\mathcal{E}}{R}\,dt$$

same amount of charge is transferred.

$$\Rightarrow \mathcal{E}_A = \mathcal{E}_B$$

Faraday's law $\quad \mathcal{E} = -\frac{d\phi}{dt}$

Paradox

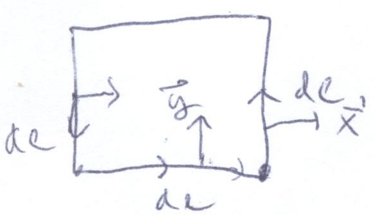
$\vec{B}$ = constant

$$\mathcal{E} = \frac{1}{c}\oint \vec{v}\times\vec{B}\cdot d\vec{l}$$

$$= \frac{1}{c}\oint \frac{d\vec{x}}{dt}\times\vec{B}\cdot d\vec{l}$$

$$= \frac{1}{c}\oint d\vec{l}\times\frac{d\vec{x}}{dt}\cdot\vec{B}$$

$$= \frac{d}{dt}\frac{1}{c}\underbrace{\oint d\vec{l}\times\vec{x}\cdot\vec{B}}_{2A}$$



wrong

solution: the contour also depends on t.

parameterication of c $\quad x(\tau),\ 0\le\tau<1$

$x(0)=x(1)$

$$\mathcal{E}=\frac{1}{c}\int_0^1 \underbrace{d\tau\,\frac{d\vec{x}}{d\tau}}_{\vec{dl}}\times\frac{d\vec{x}}{dt}\cdot\vec{B}$$

$$=\underbrace{\frac{1}{c}\frac{d}{dt}\int_0^1 d\tau\,\frac{d\vec{x}}{d\tau}\times\vec{x}\cdot\vec{B}}_{-\frac{2}{c}\frac{d\vec{A}}{dt}\cdot\vec{B}}-\underbrace{\frac{1}{c}\int_0^1 d\tau\,\frac{d\dot{\vec{x}}}{d\tau}\times\vec{x}\cdot\vec{B}}_{+\frac{1}{c}\int_0^1 d\tau\,\dot{\vec{x}}\times\frac{d\vec{x}}{d\tau}\cdot\vec{B}}$$

(see previous page)

Part Int

$$=+\frac{1}{c}\int_c \vec{v}\times\vec{dl}\cdot\vec{B}$$

$$=-\frac{1}{c}\int\vec{v}\times\vec{B}\cdot\vec{dl}$$

$$\mathcal{E}=\frac{1}{c}\oint\vec{v}\times\vec{B}\cdot dl$$

$$\Rightarrow\ -\frac{1}{c}\frac{d\Phi}{dt}\,2=\mathcal{E}\,2$$

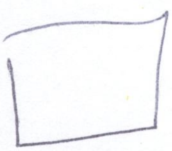
$$\Rightarrow\ \mathcal{E}=-\frac{1}{c}\frac{d\Phi}{d\tau}$$

Lecture #10

$$\mathcal{E} = -\frac{1}{c}\frac{d\phi}{dt}$$

paradox

B constant



$$\mathcal{E} = \frac{1}{c}\int \vec{v}\times\vec{B}\cdot dl$$

$$= \frac{d}{dt}\frac{1}{c}\int \vec{x}\times\vec{B}\cdot d\vec{l}$$

$$= \frac{1}{c}\frac{d}{dt}\underbrace{\int d\vec{l}\times\vec{x}}_{-2A}\cdot\vec{B}$$

missing term $-\frac{1}{c}\int \frac{d\vec{l}}{dt}\times\vec{x}\cdot\vec{B}$

$$= -\frac{1}{c}\int_0^1 d\tau\, \frac{d\dot{\vec{x}}}{d\tau}\times\vec{x}\,\vec{B}$$

$$\text{PI} = +\frac{1}{c}\int_0^1 d\tau\, \dot{\vec{x}}\times\frac{d\vec{x}}{d\tau}\cdot B$$

$$= \frac{1}{c}\int_0^1 d\tau\, \vec{B}\times\dot{\vec{x}}\,\frac{d\vec{x}}{d\tau}d\tau$$

$$= -\frac{1}{c}\int_0^1 d\tau\, \vec{v}\times\vec{B}\cdot d\vec{l}$$

$$= -\mathcal{E}$$

$$\Rightarrow \mathcal{E} = -2A\cdot B - \mathcal{E}$$

$$\Rightarrow 2\mathcal{E} = 2A\,B$$

OK

Maxwell equation

1) $\vec{\nabla} \times \vec{E} = -\frac{1}{c} \partial_t B$

2) $\vec{\nabla} \cdot \vec{E} = 4\pi \rho$

3) $\vec{\nabla} \cdot \vec{B} = 0$

4) $\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} j + \frac{1}{c} \frac{\partial \vec{E}}{\partial t}$

We will now argue that this term should be there

1) and 3) are consistent

$$\underbrace{\vec{\nabla} \cdot \vec{\nabla} \times \vec{E}}_{\shortparallel \atop 0} = -\frac{1}{c} \partial_t \vec{\nabla} \vec{B} = 0$$

Continuity eq $\partial_t \rho + \vec{\nabla} \vec{j} = 0$

$$\partial_t \rho \neq 0 \Rightarrow \vec{\nabla} \vec{j} \neq 0$$

and $\underbrace{\vec{\nabla} \cdot (\vec{\nabla} \times \vec{B})}_{\shortparallel \atop 0} = \frac{4\pi}{c} \vec{\nabla} \cdot \vec{j} \neq 0$

Maxwell solved this by adding an extra term to 4

$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} (\vec{j} + \vec{j}_D)$$

displacement current

To be consistent, we need that

$$\vec{\nabla} \cdot (\vec{j} + \vec{j}_D) = 0$$

$$\Rightarrow \vec{\nabla} \vec{j}_D = -\vec{\nabla} \vec{j} = +\frac{\partial \rho}{\partial t}$$

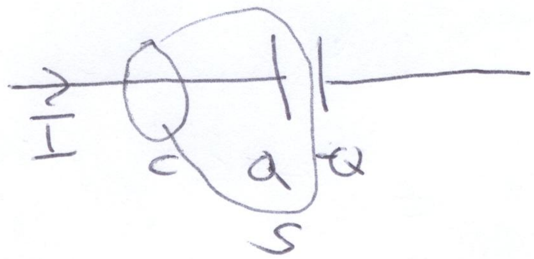
$$= \frac{1}{4\pi} \vec{\nabla} \frac{\partial \vec{E}}{\partial t}$$

$$\Rightarrow \vec{j}_D = \frac{1}{4\pi} \frac{\partial \vec{E}}{\partial t} + \vec{\nabla} \times \vec{Q}$$

The simplest choice $\vec{Q} = 0$ appeared to be the right choice.

$$\Rightarrow \text{ME 4)}: \quad \vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{j} + \frac{1}{c} \partial_t \vec{E}$$

Example of displacement current



$$\oint_C \vec{B} \cdot dl = \frac{4\pi}{c} I$$

$$\parallel$$

$$\int_S \vec{\nabla} \times \vec{B} \cdot \vec{dS}$$

Choose S through capacitor.

~~which~~ without displacement current we have that $\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{j} = 0$

with displacement current

$$\vec{\nabla} \times \vec{B} = \underbrace{\frac{4\pi}{c} j}_{=0} + \frac{1}{c} \partial_t \vec{E}$$

$$\Rightarrow \int_S \vec{\nabla} \times \vec{B} \cdot dS = \int_S \frac{1}{c} \partial_t \vec{E} \cdot dS$$

$$= \frac{1}{c} \partial_t \int_S \vec{E} \cdot dS$$

We can close S because $\vec{E} = 0$ inside C

$$= \frac{1}{c} \partial_t \int_V \underbrace{\text{div}}_{4\pi\rho} \vec{E} \cdot dV$$

$$= \frac{1}{c} 4\pi \partial_t Q = \frac{4\pi}{c} I$$ okay

Lecture #19

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RE $\nabla \times E = -\frac{1}{c}\partial_t B$ Faraday

$\vec{\nabla}\cdot\vec{B} = 0$

$\vec{\nabla}\cdot\vec{E} = 4\pi\rho$

displacement current

$\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J} + \underbrace{\frac{1}{c}\frac{\partial\vec{E}}{\partial t}}$

necessary for consistency

Today: - Magnetic energy

- momentum tensor

Faraday's law

$$\mathcal{E} = -\frac{d\phi}{dt}$$

$$\oint_C \vec{E}_{eff}\cdot d\vec{\ell} = -\frac{d}{dt}\int \vec{B}\cdot d\vec{A}$$

|| stoke

$$\int \vec{\nabla} \times \vec{E} \cdot d\vec{A}$$

True for arbitrary loops

$$\Rightarrow \quad \vec{\nabla}\times\vec{E} = -\frac{d}{dt}\vec{B}$$

differential form of Faraday's law.

Magnetic energy

a magnetic field acting on a current transfers energy to the current.
The loss of energy of the field per second is given by

$$-\frac{dW}{dt} = \int d^3r\, \vec{E}\cdot\vec{J} = \int d^3r\, \vec{E}\cdot \frac{c}{4\pi}\vec{\nabla}\times\vec{H}$$

$$= \int d^3r\, \frac{c}{4\pi}\, \vec{H}\cdot\vec{\nabla}\times\vec{E}$$

$$= -\frac{c}{4\pi}\int d^3r\, \partial_t\vec{B}\cdot\vec{H}$$

$B = \mu H$

$$= -\frac{c}{8\pi}\partial_t\int d^3r\,(\vec{B}\cdot\vec{H})$$

$$\Rightarrow W = \frac{1}{8\pi}\int d^3r\, \vec{B}\cdot\vec{H}$$

$$dW = -\sum_i q_i \vec{E}_i \cdot d\vec{s}$$

$$\frac{dW}{dt} = -\sum_i q_i \vec{E}_i \frac{d\vec{s}}{dt}$$

$$= -\int d^3r\, \vec{E}(r)\, \underbrace{\rho(r)\vec{v}}_{J}$$

We can rewrite this in terms of the vector potential:

$$W = \frac{1}{8\pi}\int d^3r\, \vec{B}\cdot\vec{H} = \frac{1}{8\pi}\int d^3r\, \vec{\nabla}\times\vec{A}\cdot\vec{H}$$

$$= \frac{1}{8\pi}\int d^3r\, \varepsilon_{ijk}\,\partial_j A_k \cdot H_i$$

$$= -\frac{1}{8\pi}\int d^3r\, \varepsilon_{ijk}\, A_k\, \partial_j H_i$$

$$= \frac{1}{8\pi}\int d^3r\, \varepsilon_{kji}\, A_k\, \partial_j H_i$$

$$= \frac{1}{8\pi}\int d^3r\, A\cdot \underbrace{\vec{\nabla}\times\vec{H}}_{\frac{4\pi}{c}\vec{j}}$$

$$W = \frac{1}{2c}\int d^3r\, \vec{A}\cdot\vec{j}$$

For a circuit this becomes $d^3r\,\vec{j} = I\,d\ell$

$$W = \frac{1}{2c}\oint \vec{A}\cdot I\,d\vec{\ell}$$

$$= \frac{1}{2c}\int_S I\,\underbrace{\vec{\nabla}\times\vec{A}}_{B}\cdot d\vec{A} = \frac{1}{2c}\, I\, \underbrace{\Phi}_{\text{flux}}$$

magnetic field depend linearly on the currents $\Rightarrow$ $\Phi_i = c\sum_j \underbrace{L_{ij}}_{\text{coefficients of inductio}} I_j$

$\Rightarrow W = \frac{1}{2}\sum I_i L_{ij} I_j$

Momentum tensor

$$\vec{F} = \frac{1}{c}\int_V d^3r\, \vec{j} \times \vec{B} \qquad \vec{j} = \sum_i q_i \vec{v}_i\, \delta(\vec{r}-\vec{r}_i)$$

force of field on current

$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c}\vec{j}$$

$$\Rightarrow F = \frac{1}{4\pi}\int_V d^3r\, \underbrace{(\vec{\nabla}\times\vec{B})\times\vec{B}}_{-\vec{B}\times(\vec{\nabla}\times B)}$$

$$F_k = -\frac{1}{4\pi}\int_V d^3r\; B_i\partial_k B_i - B_i \partial_i B_k$$

$$= -\frac{1}{4\pi}\int_V d^3r\; \frac{1}{2}\partial_k \vec{B}^2 - \partial_i(B_i B_k)$$

note that $\partial_i B_i = 0$

$$= \frac{1}{4\pi}\int_V d^3r\; \partial_i\left(B_i B_k - \frac{1}{2}\delta_{ik}\vec{B}^2\right)$$

$$= \int_{\partial V} da\, n_i\, T_{ik}$$

momentum tensor $T_{ik} = \frac{1}{4\pi}\left(B_i B_k - \frac{1}{2}\delta_{ik}\vec{B}^2\right)$

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V

Force on charges and current

$$\vec{F} = \int d^3r \left(\rho \vec{E} + \frac{1}{c} \vec{j} \times \vec{B} \right)$$

$$\rho = \frac{1}{4\pi} \vec{\nabla} \vec{E} \qquad \vec{j} = \frac{c}{4\pi}(\vec{\nabla} \times \vec{B}) - \frac{1}{4\pi}\partial_t \vec{E}$$

$$\Rightarrow F_i = \frac{1}{4\pi} \int d^3r \left(E_i \partial_k E_k - \frac{1}{c}\partial_t (\vec{E} \times B)_i + ((\vec{\nabla} \times \vec{B}) \times \vec{B})_i \right)$$

$$E_i \partial_k E_k = \partial_k (E_i E_k) + E_k \partial_i E_k - E_k \partial_k E_i - E_k \partial_i E_k$$

$$= \partial_k \left(E_i E_k - \frac{1}{2} \delta_{ik} \vec{E}^2 \right) + E_k (\partial_i E_k - \partial_k E_i)$$

$$(\vec{E} \times (\vec{\nabla} \times \vec{E}))_i = E_k \, \varepsilon_{ikp} \varepsilon_{prs} \partial_r E_s = E_k \partial_i E_k - E_k \partial_k E_i$$

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\partial_t B$$

$$((\vec{\nabla} \times \vec{B}) \times \vec{B})_i = -\vec{B} \times (\vec{\nabla} \times \vec{B}) = -B_k \partial_i B_k + B_k \partial_k B_i$$

$$= \partial_k \left(B_k B_i - \frac{1}{2} \delta_{ik} \vec{B}^2 \right)$$

$$\Rightarrow F_i = -\int \frac{d^3r}{4\pi c} \partial_t (\vec{E} \times \vec{B}) + \int d^3r \, \partial_k T_{ik}$$

$$T_{ik} = E_i E_k - \frac{1}{2} \delta_{ik} \vec{E}^2 + \dots \delta_{ik} \vec{B}^2$$

$$F_i = -\int \frac{d^3r}{4\pi c}\, \partial_t (\vec{E}\times\vec{B})_i + \int_{\partial V} ds\, n_k T_{ki}$$

$$\underbrace{F_i}_{=\frac{d\vec{P}_{mech}}{dt}} \qquad \underbrace{-\int \frac{d^3r}{4\pi c}\, \partial_t (\vec{E}\times\vec{B})_i}_{-\frac{d}{dt}\vec{P}_{field}} \qquad \underbrace{\int_{\partial V} ds\, n_k T_{ki}}_{\text{momentum flow through surface}}$$

$$\Rightarrow \frac{d}{dt}\left(\vec{P}_{mechanical} + \vec{P}_{field}\right) = \int_{\partial V} ds\, n_k T_{ki}$$

$\Rightarrow$ momentum of the em field is

$$\vec{P} = \frac{1}{4\pi c}\int d^3r\, (\vec{E}\times\vec{B})$$

The angular momentum of the field is $\vec{L} = \frac{1}{4\pi c}\int d^3r\, \vec{r}\times(\vec{E}\times\vec{B})$

Lecture #20

$$-\frac{dW}{dt} = \int d^3r\, \vec{E}\cdot\vec{J}$$

$$\Rightarrow W = \frac{1}{8\pi}\int d^3r\, \vec{B}\cdot\vec{H}$$

$$= \frac{1}{2c}\int d^3r\, \vec{A}\cdot\vec{J}$$

$$\phi_i = c\sum_j L_{ij} I_j$$

$$W = \frac{1}{2}\sum_{ij} I_i L_{ij} I_j$$

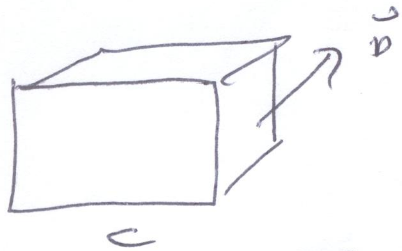
$$F_k = \int_{\partial V} da\, n_i T_{ik}$$

$$T_{ik} = \frac{1}{4\pi}\left(B_i B_k - \frac{1}{2}\delta_{ik} \vec{B}^2\right.$$

<u>Today</u> Force on charges or currents

momentum density

$$\vec{p} = \frac{\vec{E} \times \vec{B}}{4\pi c}$$



momentum flow per unit area per second is

$$c . \vec{p} . \vec{n} = \frac{\vec{E} \times \vec{B}}{4\pi} . \vec{n}$$

energy $E = pc$

$\Rightarrow$ energy flow per second per unit area

is $c\frac{\vec{E} \times \vec{B}}{4\pi} . \vec{n}$

$\vec{P} \equiv c\frac{\vec{E} \times \vec{B}}{4\pi}$ is known as the

Poynting vector.

Energy conservation

$$E\cdot\frac{\partial D}{\partial t} + H\cdot\frac{\partial B}{\partial t} = E\cdot(c\nabla\times\vec{H} - 4\pi\vec{j})$$
$$- Hc\cdot\vec{\nabla}\times\vec{E}$$

$-\frac{dW}{dt}$

$$\Rightarrow \underbrace{-\frac{1}{4\pi}\int d^3r\, \vec{E}\cdot\frac{\partial\vec{D}}{\partial t} + \vec{H}\cdot\frac{\partial B}{\partial t}}_{\text{decrease of field energy}} = \int d^3r\, \vec{j}\cdot\vec{E}$$

$$+\frac{c}{4\pi}\int d^3r\, \underbrace{\vec{E}\cdot\vec{\nabla}\times\vec{H} - \vec{H}\cdot\vec{\nabla}\times}$$

work done

$$= \frac{c}{4\pi}\int d^3r\, \vec{\nabla}\cdot(\vec{E}\times\vec{H})$$
$$\shortparallel$$

energy flowing out $\rightarrow$ $\frac{c}{4\pi}\int d\vec{S}\cdot\vec{E}\times\vec{H}$

$$\vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{H}$$

$$\vec{E}\cdot\vec{\nabla}\times H - H\cdot\vec{\nabla}\times\vec{E}$$

$$E_i\varepsilon_{ijk}\partial_j H_k - H_i\varepsilon_{ijk}\partial_j E_k$$

$$\varepsilon_{jik}(-\partial_k H_k E_i + H_i\partial_j E_k) = \varepsilon_{jik}(\partial_j H_k E$$
$$- H_k\partial_j E$$