1a

Lecture 1

Maxwell equation

a) Coulomb's law

b) Electric field

Maxwell eqs.

$$\vec{\nabla}\vec{E} = 4\pi\rho$$
$$\vec{\nabla}\vec{B} = 0$$
$$\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t}$$
$$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$$

The goal of this class i to unstand these equations.

In covariant form they are very simple

$$\underbrace{\partial_\mu}_{\text{4 derivative}} \underbrace{F_{\mu\nu}}_{\text{em field tensor}} = \underbrace{j_\nu}_{\text{4 current.}}$$

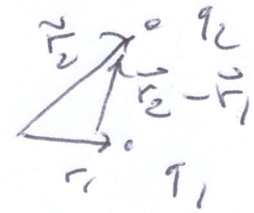
ME are relativistically correct but not quantum mechanically. The quantum theory is Quantum Elektro Dynamics QED.

The Maxwell equations simplify when the time derivatives vanish. Then the equations for $\vec{E}$ and $\vec{B}$ decouple, and we have electrostatics and magnetostatics.

a)

For electrostatics, all you need is <u>Coulomb's law</u>

$$\vec{F}_{21} = q_1 q_2 \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|^3}$$



- $\vec{F}_{12} = -F_{21}$

- forces are additive

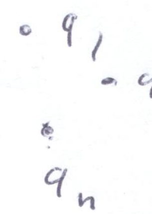
· in electrostatics you can solve all problems using Coulomb's law only.

b) <u>Electric field</u>

This is the force per unit charge

$$\vec{E} = \frac{\vec{F}}{q}$$

$$\vec{E}(\vec{r}) = \sum_i q_i \frac{(\vec{r} - \vec{r}_i)}{|\vec{r} - \vec{r}_i|^3}$$



charge density $\quad \rho(r) \equiv \sum_i q_i \delta^3(r - \vec{r}_i)$

$$\Rightarrow \quad E(\vec{r}) = \int d^3r' \, \rho(r') \frac{(\vec{r} - r')}{|\vec{r} - r'|^3}$$

Lecture # 2

3a

$$\vec{E} = \frac{\vec{F}}{q} \qquad \vec{E} = \sum q_i \frac{(\vec{r} - \vec{r}_i)}{|\vec{r} - \vec{r}_i|^3}$$

charge density $\rho(\vec{r}) = \sum_i q_i \, \delta(\vec{r} - \vec{r}_i)$

$$E(r) = \int d^3r' \, \rho(r') \frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}$$

Today
Potential
Gauss law
Poisson equation
Uniqueness theorem

c) Potential

$$\frac{\vec{r} - \vec{r}_i}{|\vec{r} - \vec{r}_i|^3} = -\vec{\nabla} \frac{1}{|\vec{r} - \vec{r}_i|}$$

$$= -\partial_{x_k} \frac{1}{\sqrt{(x_1 - x_{1i})^2 + (x_2 - x_{2i})^2 + (x_3 - x_{3i})^2}}$$

$$\Rightarrow \vec{E} = \vec{\nabla} \cdot \underbrace{\sum_i \frac{q_i}{|\vec{r} - \vec{r}_i|}}_{\text{potential } \phi(\vec{r})}$$

$$\phi(\vec{r}) = \sum_i \frac{q_i}{|\vec{r} - \vec{r}_i|} = \int \rho(r') \frac{1}{|r - r'|}$$

Since $\vec{E} = -\vec{\nabla} \cdot \phi$

we have that $\vec{\nabla} \times \vec{E} = -\vec{\nabla} \times \vec{\nabla} \phi = 0$

so this ME follows from Coulomb's law

we can also invert the equation

$$\phi(\vec{r}) = -\int_{\vec{r}_0}^{\vec{r}} \vec{E}(\vec{s})\, d\vec{s}$$

to be unique we need

$$-\int_{\vec{r}_0, 1}^{\vec{r}} \vec{E} \cdot d\vec{s} = \int_{\vec{r}_0, 2}^{\vec{r}} E \cdot d\vec{s}$$

Lecture 3

$$\phi = \int d^3r' \frac{\rho(r')}{|\vec{r}-\vec{r}'|}$$

~~Cauchy the~~ Stokes theorem

$$\oint_{C=\partial A} \vec{V}\cdot d\vec{s} = \int \vec{\nabla}\times\vec{V}\cdot d\vec{A}$$

Today $\phi = -\int_{r_0}^{\vec{r}} \vec{E}\cdot ds$ is path independent

Gauss Law $\vec{\nabla}\cdot\vec{E} = 4\pi\rho$

Poisson eq. $\vec{\nabla}^2\phi = -4\pi\rho$

Laplace eq $\vec{\nabla}^2\phi = 0$

This is the case if

$$\oint \vec{E} \cdot d\vec{s} = 0$$

Stokes theorem

$$\oint_{\partial C} \vec{E} \cdot ds = \int_C \underbrace{\vec{\nabla} \times \vec{E}}_{=0} \cdot d\vec{A}$$

$$= 0$$

d) <u>Gauss law</u>

We have that $\vec{E} = \int \frac{\rho(r') (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}$

now we want to invert this equation an express ρ in $\vec{E}$.

We first prove

$$\vec{\nabla}_r \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} = 4\pi \delta^3 (r - \vec{r}')$$

$$\parallel$$

$$- \vec{\nabla}_r \vec{\nabla}_r \frac{1}{|\vec{r} - \vec{r}'|}$$

$\vec{\nabla}_r^2$ is known as the Laplacian

$$\vec{\nabla}_r = \partial_x^2 + \partial_y^2 + \partial_z^2$$

$\vec{r}'$ does not matter, so we choose $\vec{r}' = 0$

We have to prove that

$$\vec{\nabla}_r^2 \frac{1}{|\vec{r}|} = -4\pi\,\delta^3(\vec{r})$$

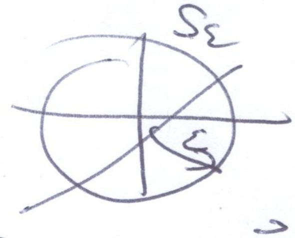
ok if $\vec{r} \neq 0$

$$\partial_x^2 \frac{1}{\sqrt{x^2+y^2+z^2}} = \partial_x \frac{-x}{(x^2+y^2+z^2)^{3/2}} =$$

$$\frac{-1}{(x^2+y^2+z^2)^{3/2}} + \frac{3x^2}{(x^2+y^2+z^2)^{5/2}}$$

$$\Rightarrow (\partial_x^2 + \partial_y^2 + \partial_z^2)\frac{1}{\sqrt{x^2+y^2+z^2}} = 0$$

To prove that is is also corcet around zero we put a small sphere around zero and integrate about it



$$\int_{S_\varepsilon} d^3r\, \vec{\nabla}_r \vec{\nabla}_r \frac{1}{|\vec{r}|} \underset{\text{Gauss theorem}}{=} \int_{\partial S_\varepsilon} d\vec{a} \cdot \underbrace{\vec{\nabla}_r \frac{1}{r}}_{-\frac{\vec{r}}{r^3}}$$

$$d\vec{a} = r^2 \hat{r}\, d\Omega = \hat{r} \sin\theta\, d\theta\, d\varphi$$

$$= \int_{\partial S_\varepsilon} r^2 d\Omega \; \frac{-1}{r^2} = -\int d\Omega = -4\pi$$

The rhs just gives -4π $\Rightarrow$ q.e.d.

We have shown that

$$\vec{\nabla}^2 \frac{1}{|\vec{r}-\vec{r}'|} = -4\pi\,\delta^3 \vec{r}$$

$$\Rightarrow \vec{\nabla}\vec{E} = -\int d^3\vec{r}' \left(\vec{\nabla}\,\vec{\nabla}\frac{1}{|r-r'|}\right)\rho(\vec{r}')$$

$$= -\int d^3\vec{r}'\,\rho(\vec{r}')\,(-4\pi)\,\delta^3(r'-\vec{r})$$

$$= +4\pi\,\rho(\vec{r})$$

<u>Gauss law</u>