Electromagnetism 1

Radiation from a relativistic electron

Consider a relativistic electron (of charge e) traveling with an initial speed v_o along the z-axis. At time $t = 0$ it slows down to a stop over a time τ while moving along the z-axis

$$v(t) = v_o \left(1 - \frac{t}{\tau}\right), \qquad 0 \le t \le \tau\,. \tag{1}$$

Recall that the electric field in the far field radiated from a point charge following a trajectory with position $\boldsymbol{x}(t)$, and velocity $\boldsymbol{v}(T) = \boldsymbol{x}'(t)$ is

$$\boldsymbol{E}_{\rm rad}(t, \boldsymbol{r}) = \frac{e}{4\pi c^2} \left[\frac{\boldsymbol{n} \times (\boldsymbol{n} - \boldsymbol{\beta}) \times \boldsymbol{a}}{R\,(1 - \boldsymbol{n}\cdot\boldsymbol{\beta})^3}\right]_{\rm ret}, \tag{2}$$

where all quantities in square brackets are evaluated at the *retarded time*, $T(t, \boldsymbol{r})$ (which you will define below). The other symbols are defined as $\boldsymbol{n} \equiv (\boldsymbol{r} - \boldsymbol{x}(T))/|\boldsymbol{r} - \boldsymbol{x}(T)|$, $R \equiv |\boldsymbol{r} - \boldsymbol{x}(T)|$, and $\boldsymbol{\beta} = \boldsymbol{v}/c$.

(a) (3 points) Define the retarded time and compute the derivatives $\partial T/\partial t$ and $\partial T/\partial \boldsymbol{r}^i$

(b) (3 points) The radiation field $\boldsymbol{E}_{\rm rad}$ is derived from the *Liénard-Wiechert* potentials

$$\varphi(t, \boldsymbol{r}) = \frac{e}{4\pi}\left[\frac{1}{R(1 - \boldsymbol{n}\cdot\boldsymbol{\beta})}\right]_{\rm ret}, \tag{3}$$

$$\boldsymbol{A}(t, \boldsymbol{r}) = \frac{e}{4\pi c}\left[\frac{\boldsymbol{v}}{R(1 - \boldsymbol{n}\cdot\boldsymbol{\beta})}\right]_{\rm ret}. \tag{4}$$

Using far field approximations, show that the Lorenz gauge condition is satisfied by these potentials.

(c) (6 points) For the decelerating electron described above, compute:

(i) the energy radiated per solid angle per *retarded time*.

(ii) the energy radiated per solid angle *per time*.

Describe in what physical situations you would be interested in (i) and (ii) respectively. Use no more than two sentences to describe each case.

(d) (4 points) Now consider a relativistic electron with initial energy of 1 GeV.

Examining your results of part (c), you should find that at $t = 0$ the radiation is initially emitted (predominantly) at a characteristic angle. Give an order of magnitude estimate for this angle. Explain your estimate by pointing to specific terms in your formulas from part (c).

(e) (4 points) Determine the total energy per solid angle emitted as the electron decelerates to a stop.

Solution

(a) The retarded time is the time that light was emitted at the source such that it arrives at space-time observation point $(t, \boldsymbol{r})$. It satisfies the implicit equation

$$t - T = |\boldsymbol{r} - \boldsymbol{x}(T)|/c\,. \tag{5}$$

Differentiating

$$1 - \frac{\partial T}{\partial t} = - \frac{(\boldsymbol{r} - \boldsymbol{x}(T))^\ell}{|\boldsymbol{r} - \boldsymbol{x}(T)|} v_\ell(T)/c \frac{\partial T}{\partial t}\,, \tag{6}$$

$$1 - \frac{\partial T}{\partial t} = - \boldsymbol{n} \cdot \beta(T) \frac{\partial T}{\partial t}\,. \tag{7}$$

Thus

$$\frac{\partial T}{\partial t} = \frac{1}{1 - \boldsymbol{n} \cdot \boldsymbol{\beta}(T)}\,. \tag{8}$$

Similarly,

$$-\frac{\partial T}{\partial r^k} = \frac{(\boldsymbol{r} - \boldsymbol{x}(T))^\ell}{|\boldsymbol{r} - \boldsymbol{x}(T)|} \left(\delta_{\ell k} - \frac{v_o(T)_\ell}{c} \frac{\partial T}{\partial r^k} \right)\,. \tag{9}$$

Thus

$$\frac{\partial T}{\partial r^k} = \frac{-n_k}{(1 - \boldsymbol{n} \cdot \boldsymbol{\beta}(T))}\,. \tag{10}$$

(b) The Lorenz gauge condition reads

$$\frac{1}{c} \partial_t \varphi + \partial_i A^i = 0\,. \tag{11}$$

In the far field we neglect differentiating $1/R$ and $\boldsymbol{n}$ which lead to subleading terms in $1/R$. Then in the far field we differentiate

$$\frac{1}{c} \partial_t \varphi = \frac{e}{4\pi R c^2} \frac{\boldsymbol{n} \cdot \boldsymbol{a}}{(1 - \boldsymbol{n} \cdot \boldsymbol{\beta})^2} \frac{\partial T}{\partial t}\,, \tag{12}$$

$$= \frac{e}{4\pi R c^2} \frac{\boldsymbol{n} \cdot \boldsymbol{a}}{(1 - \boldsymbol{n} \cdot \boldsymbol{\beta})^3}\,. \tag{13}$$

Similarly,

$$\partial_i \boldsymbol{A}^i = \frac{e}{4\pi R c^2} \left[\frac{a^i}{(1 - \boldsymbol{n} \cdot \boldsymbol{\beta})} \frac{\partial T}{\partial r^i} + \frac{\beta^i}{(1 - \boldsymbol{n} \cdot \boldsymbol{\beta})^2} (\boldsymbol{n} \cdot \boldsymbol{a}) \frac{\partial T}{\partial r^i} \right]\,, \tag{14}$$

$$= \frac{e}{4\pi R c^2} \left[\frac{-\boldsymbol{n} \cdot \boldsymbol{a}}{(1 - \boldsymbol{n} \cdot \boldsymbol{\beta})^2} + \frac{-\boldsymbol{n} \cdot \beta}{(1 - \boldsymbol{n} \cdot \boldsymbol{\beta})^3} (\boldsymbol{n} \cdot \boldsymbol{a}) \right]\,, \tag{15}$$

$$= \frac{e}{4\pi R c^2} \left[\frac{-\boldsymbol{n} \cdot \boldsymbol{a}}{(1 - \boldsymbol{n} \cdot \boldsymbol{\beta})^3} \right]\,. \tag{16}$$

So we verify that

$$\frac{1}{c} \partial_t \varphi + \partial_i \boldsymbol{A}^i = 0\,. \tag{17}$$

(c) In this case $\boldsymbol{\beta} \times \boldsymbol{a} = 0$, $|\boldsymbol{n} \times \boldsymbol{n} \times \boldsymbol{a}| = a\sin(\theta)$, and thus the magnitude of $\boldsymbol{E}$ is

$$E = \frac{e}{4\pi Rc^2} \frac{a\sin\theta}{(1-\beta(T)\cos\theta)^3} \tag{18}$$

So the energy per time per solid angle

$$\frac{dW}{dtd\Omega} = \lim_{r\to\infty} c|r\boldsymbol{E}|^2 \tag{19}$$

$$= \frac{e}{(4\pi)^2 c^3} \frac{a^2\sin^2\theta}{(1-\beta(T)\cos\theta)^6} \tag{20}$$

where $a = v_o/\tau$, and $\beta(T) = \beta_o(1-T/\tau)$. The energy *per retarded time* per solid angle is

$$\frac{dW}{dTd\Omega} = \frac{dW}{dtd\Omega}\frac{dt}{dT} \tag{21}$$

$$= \frac{e^2}{(4\pi)^2 c^3} \frac{a^2\sin^2\theta}{(1-\beta(T)\cos\theta)^5} \tag{22}$$

The energy per time is useful if you want to know whether a remote detector will burn up. The energy per retarded time is useful if you want to calculate how much energy is lost to radiation over a given element of a particles trajectory, $d\boldsymbol{x} = \boldsymbol{v}(T)dT$.

(d) We see that the denominator function, $1-\beta_o\cos\theta$, is approaching zero at small angle since $\beta_o \simeq 1$. Expanding $\beta_o \simeq 1 - \frac{1}{2\gamma^2}$ and $\cos\theta \simeq 1 - \frac{\theta^2}{2}$,

$$\frac{1}{1-\boldsymbol{n}\cdot\boldsymbol{\beta}} \simeq \frac{1}{\frac{1}{2\gamma_o^2} + \frac{\theta^2}{2}} = \frac{2\gamma_o^2}{1+(\gamma_o\theta)^2} . \tag{23}$$

So the characteristic angle is $\theta \sim 1/\gamma_o$. For a 1 GeV electron, $\gamma \simeq E/m_e c^2 \sim 2000$. So $\theta \sim 1/2000$.

(e) The total energy is

$$\frac{dW}{d\Omega} = \int_0^\tau dT \frac{dW}{dTd\Omega} . \tag{24}$$

So with the result of Eq. 21 we have

$$\frac{dW}{d\Omega} = \frac{e^2}{(4\pi)^2 c^3}(a^2\sin^2\theta)\int_0^\tau dT \frac{1}{(1-\beta_o(1-\frac{T}{\tau})\cos\theta)^5} , \tag{25}$$

$$= \frac{e^2}{(4\pi)^2 c^3} \frac{\tau(a^2\sin^2\theta)}{4\beta_o\cos\theta} \left[\frac{-1}{(1-\beta_o(1-\frac{T}{\tau})\cos\theta)^4}\right]_0^\tau , \tag{26}$$

$$= \frac{e^2}{(4\pi)^2 c^3} \frac{\tau(a^2\sin^2\theta)}{4\beta_o\cos\theta} \left[\frac{1}{(1-\beta_o\cos\theta)^4} - 1\right] . \tag{27}$$

In the ultra relativistic limit we have

$$\frac{1}{1-\beta_o\cos\theta} \simeq \frac{1}{\frac{1}{2\gamma_o^2}+\frac{\theta^2}{2}} = \frac{2\gamma_o^2}{1+(\gamma_o\theta)^2}\,, \tag{28}$$

and thus

$$\frac{dW}{d\Omega} \simeq \frac{e^2a^2\tau}{(4\pi)^2c^3}\,4\gamma_o^2\left[\frac{(\gamma_o\theta)^2}{(1+(\gamma_o\theta)^2)^4}\right]\,. \tag{29}$$

Electromagnetism 2

Induction and the energy in static magnetic fields

Consider a closed circuit of wire formed into a circular coil of n turns with radius a, resistance R, and self-inductance L. The coil rotates around the z-axis in a uniform magnetic field H directed along the x-axis (see below).

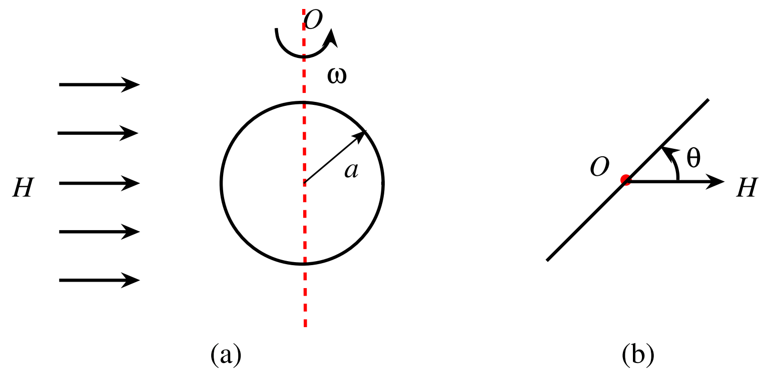


Figure 1: (a) side view; (b) top view.

a) (6 points) Find the current in the coil as a function of θ for rotation at a constant angular velocity ω. Here $\theta(t) = \omega t$ is the angle between the plane of the coil and H (the x-axis).

b) (4 points) Find the externally applied torque that is needed to maintain the coil's uniform rotation.

c) Because of the time-dependent currents induced in the coil, electromagnetic waves are radiated. Briefly answer the following questions:

(i) (2 points) What is the frequency of the radiation? Explain.

(ii) (2 points) What is the the polarization of the radiated waves propagating along the positive z-axis? Explain.

d) (6 points) Compute the total power radiated by the rotating coil of wire.

Note: in all parts you should assume that all transient effects have died away.

Solution:

a) Let I be the current in the coil, we have

$$\mathcal{E} = IR = -L\frac{dI}{dt} - \frac{1}{c}\frac{\partial \Phi_H}{\partial t}\,, \tag{1}$$

where the flux is given by $\Phi_H = \pi a^2\, n\, H \sin\theta(t)$ with $\theta(t) = \omega t$. With these phase conventions, the area vector of the loop points in the negative $\hat{\boldsymbol{y}}$ direction at $t = 0$ and in the $\hat{\boldsymbol{x}}$ direction at $\omega t = \pi/2$. Thus the circulation of a positive current at $t = 0$ is specified with the right hand rule with the thumb pointing in the negative $\hat{\boldsymbol{y}}$ direction.

From Eq. (1), we have the differential equation for the current,

$$L\,\frac{dI}{dt} + R\,I = -\frac{\pi a^2}{c} n\,H\,\omega\,\cos(\omega t)\,, \tag{2}$$

and corresponding solution as

$$\begin{aligned} I(t) &= -\frac{\pi a^2\, n\, H\, \omega}{c}\,\frac{1}{2}\left[\frac{e^{i\omega t}}{R + i\omega L} + \frac{e^{-i\omega t}}{R - i\omega L}\right] \\ &= -\frac{\pi a^2\, n\, H}{c}\frac{\omega}{\sqrt{R^2 + \omega^2 L^2}}\,\cos(\omega t + \phi)\,, \end{aligned} \tag{3}$$

where the phase $\phi = \tan^{-1}(-\omega L/R)$.

b) The rotating coil has a magnetic dipole moment, $\boldsymbol{\mu}(t) = I(t)\vec{A}(t)/c$. With the conventions of the previous part we have

$$\boldsymbol{\mu}(t) = m_o \cos(\omega t + \phi)\,\left(-\sin(\omega t)\hat{\boldsymbol{x}} + \cos(\omega t)\hat{\boldsymbol{y}}\right)\,. \tag{4}$$

where

$$m_o \equiv \left(\frac{\pi a^2 n}{c}\right)^2 \frac{\omega}{\sqrt{R^2 + \omega^2 L^2}} H\,. \tag{5}$$

The torque on the loop is $\boldsymbol{\mu} \times \mathbf{H}$, and an external torque of $\boldsymbol{\tau}_{\rm ext} = -\boldsymbol{\mu} \times \mathbf{H}$ is needed to keep the coil rotating at a constant angular velocity is (with $\mathbf{H} = H\hat{\boldsymbol{x}}$) :

$$\boldsymbol{\tau}_{\rm ext}(t) = m_o H \cos(\omega t + \phi)\cos(\omega t)\,\hat{\boldsymbol{z}} \tag{6}$$

c)

(i) From the solution in part (b), we see that the current induces a magnetic moment which is oscillating in time. Writing the magnetic moment in complex notation (with the understanding that the physical quantity corresponds to the real part) we see that

$$\boldsymbol{\mu}(t) = \frac{1}{2} m_o e^{-2i\omega t - i\phi}\,(-i\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}}) + \text{const}\,, \tag{7}$$

where we have neglected a constant vector,

$$\text{const} = \frac{m_o}{2} e^{i\phi} \left(-i\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}} \right) , \tag{8}$$

which does not contribute to the radiation. We see that the frequency of the radiation is 2ω.

(ii) The polarization must be transverse to the $\hat{\boldsymbol{z}}$, *i.e.* $\hat{\boldsymbol{z}} \cdot \boldsymbol{\epsilon}^* = 0$. Given the fact that the rotating magnetic moment does not prefer the x or y axes, the only two possible choices are right or left handed circularly polarized light. The magnetic moment is rotating around the z axis according to the right hand rule and the magnetic field will follow this orientation. Thus the light traveling on the z-axis will be circularly polarized with positive helicity (*i.e.* right handed).

d) A general formula for magnetic dipole radiation for a harmonic dipole moment $\boldsymbol{\mu}(t) = \boldsymbol{m}e^{-i\omega t}$ is

$$P = \frac{1}{4\pi} |\boldsymbol{m}|^2 \frac{\omega^4}{3c^3} \,. \tag{9}$$

with $\boldsymbol{m}$ a complex vector.

Adapting this formula to the problem at hand we have the replacements

$$\omega \to 2\omega \qquad \boldsymbol{m} \to \frac{m_o}{2} (-i\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}}) \,, \tag{10}$$

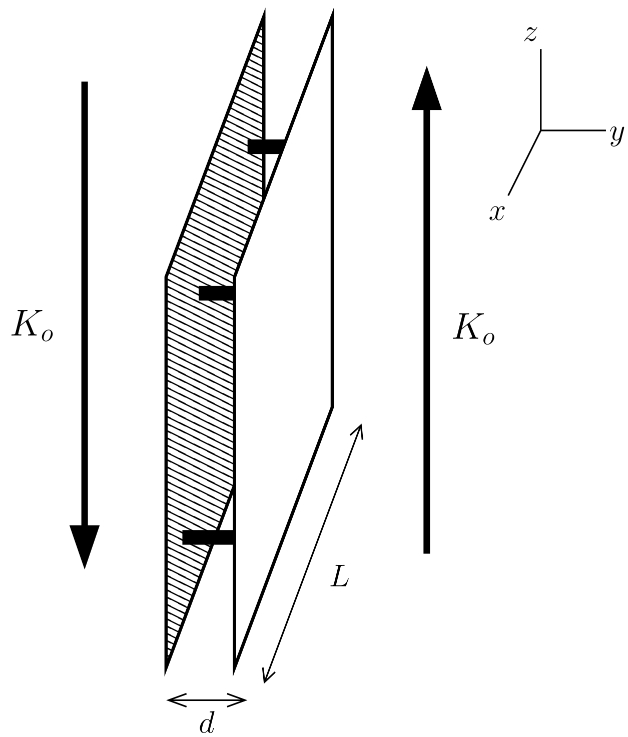
yielding

$$P = \frac{2}{\pi} m_o^2 \frac{\omega^4}{3c^3} \tag{11}$$

Electromagnetism 3

Two current sheets under Lorentz boosts

Consider two large square sheets of conducting material (with sides of length L separated by a distance d, $d \ll L$) each carrying a uniform surface current of magnitude K_o. (The total current in each sheet is $I_o = K_o L$.) The current flows up the right sheet and returns down the left sheet. The mass of the sheets is negligible. The sheets are mechanically supported by four electrically neutral columns of mass $M_{\rm col}$ and cross sectional area $A_{\rm col}$ (three shown). Neglect all fringing fields.



(a) (3 points) Write down the electromagnetic stress tensor $\Theta^{\mu\nu}_{\rm em}$ covariantly in terms of $F^{\mu\nu}$ and compute all non-vanishing components of $F^{\mu\nu}$ and $\Theta^{\mu\nu}_{\rm em}$ both in between and outside of the two sheets.

(b) (1 point) Compute the total rest energy of the system (or $M_{\rm tot}c^2$) including the contribution from the electromagnetic energy.

(c) (3 points) Determine the electromagnetic force per area on the current sheets (magnitude and direction) and the components of the mechanical stress tensor in the columns, $\Theta^{00}_{\rm mech}$ and $\Theta^{yy}_{\rm mech}$ (use the coordinates system in the figure). You can assume that the stress is constant across the cross sectional area of the columns.

(d) (6 points) Now consider the system according to an observer moving relativistically with velocity $\beta = v/c$ up the z-axis.

(i) Determine the electric and magnetic fields (magnitudes and directions) using a Lorentz transformation. Check that the direction of the Poynting vector measured by this observer is consistent with physical intuition.

(ii) Determine the charge and current densities in the sheets according to this observer. Are your charges and currents consistent with the fields computed in the first part of (d)? Explain.

(e) (7 points) Now consider the system according to an observer moving relativistically with velocity $\beta = v/z$ to the *right* along the y-axis (use the coordinate system shown in the figure).

(i) Determine the total mechanical energy in the columns according to this observer.

(ii) Determine the total electromagnetic energy according to this observer.

(iii) Determine the total energy of this configuration. Is your result for the total energy consistent with part (b)? Explain.

Comment: There is of course stress in the sheets. But, since it does not have a yy component the stress in the sheets can be neglected in this problem.

Solution

(a) The stress tensor is

$$\Theta^{\mu\nu}_{\rm em} = F^{\mu\alpha}F^{\nu}_{\ \alpha} + \eta^{\mu\nu}\left(-\tfrac{1}{4}F^2\right) \, . \tag{1}$$

The only nonzero field component is the x component of the magnetic field. Using boundary conditions or Ampère's rule

$$\boldsymbol{n} \times (\boldsymbol{B}_{\rm out} - \boldsymbol{B}_{\rm in}) = \frac{K_o}{c}\,\hat{\boldsymbol{z}}\,, \tag{2}$$

we find

$$B_x = \frac{K_o}{c}\,, \tag{3}$$

in between the sheets and zero outside the sheets. Thus only non-zero component of $F^{\mu\nu}$ is

$$F^{23} = \frac{K_o}{c}\,. \tag{4}$$

The non-zero temporal components of $\Theta^{\mu\nu}_{\rm em}$ are

$$\Theta^{00}_{\rm em} = \tfrac{1}{2}B^2 = \tfrac{1}{2}(K_o/c)^2 \tag{5}$$

The spatial components of $\Theta^{\mu\nu}$ are expressed in terms of the magnetic fields as:

$$\Theta^{ij}_{\rm em} = -B^iB^j + \frac{\delta^{ij}}{2}B^2\,. \tag{6}$$

So the non-zero spatial components are

$$-\Theta^{xx}_{\rm em} = \Theta^{yy}_{\rm em} = \Theta^{zz}_{\rm em} = \tfrac{1}{2}(K_o/c)^2\,. \tag{7}$$

(b) The total energy is a sum of the rest energy of the columns and the electromagnetic energy (the energy density in Eq. (5) times the volume)

$$M_{\rm tot}c^2 = 4M_{\rm col}c^2 + \left[L^2 d\,\tfrac{1}{2}(K_o/c)^2\right] \tag{8}$$

(c) The force per area on the sheets is the discontinuity in the stress tensor. For a normal n_i pointing from "in" to "out" the force is

$$\frac{F^j}{A} = -n_i(\Theta^{ij}_{\rm out} - \Theta^{ij}_{\rm in})\,, \tag{9}$$

and therefore, for the problem at hand, the electromagnetic force per area is

$$\left(\frac{F^y}{A}\right) = \Theta^{yy}_{\rm em} = \tfrac{1}{2}(K_o/c)^2 \,. \tag{10}$$

This is the force per area on the right sheet and is directed outward. The force per area on the left sheet is also directed outward

$$\left(\frac{F^y}{A}\right) = -\tfrac{1}{2}(K_o/c)^2 \,. \tag{11}$$

Note: the is exactly *half* of what would get for surface current in a uniform magnetic field of K_o/c (the field in between the sheets). Indeed, the force on the currents in the right sheet can be interpreted as arising from the fields generated by the currents in the left sheet. This left-sheet-generated field strength is $\frac{1}{2}K_o/c$.

The net total force on the sheets is zero (otherwise the configuration would not be stable). Thus, the electromagnetic force is balanced by the mechanical forces in the columns. The mechanical force per area in the four columns is therefore

$$\Theta^{yy}_{\rm mech} = -\frac{\frac{1}{2}L^2(K_o/c)^2}{4A_{\rm col}} \,, \tag{12}$$

where the factor of four accounts for the four columns. The mechanical energy density in the columns is

$$\Theta^{00}_{\rm mech} = \frac{M_{\rm col}c^2}{A_{\rm col}d} \,. \tag{13}$$

(d) Now we will boost the configuration. $\boldsymbol{\beta}$ is the velocity of the new observer, $\boldsymbol{\beta} = \beta\hat{\boldsymbol{z}}$.

(i) To determine the boosted fields we note the transformation rules

$$\underline{E}_\parallel = E_\parallel \,, \tag{14}$$

$$\underline{\boldsymbol{E}}_\perp = \gamma\boldsymbol{E}_\perp + \gamma\boldsymbol{\beta}\times\boldsymbol{B}_\perp \,, \tag{15}$$

and

$$\underline{B}_\parallel = B_\parallel \,, \tag{16}$$

$$\underline{\boldsymbol{B}}_\perp = \gamma\boldsymbol{B}_\perp - \gamma\boldsymbol{\beta}\times\boldsymbol{E}_\perp \,, \tag{17}$$

and thus in this case we have

$$\underline{E}^y = \gamma\beta(K_o/c) \,, \tag{18}$$

$$\underline{B}^x = \gamma(K_o/c) \,. \tag{19}$$

The direction of $\underline{\boldsymbol{E}}\times\underline{\boldsymbol{B}}$ is in the negative z direction. This makes sense – according to an observer moving the positive z direction the fields have a net momentum in the negative z direction.

(ii) To boost the currents we first record the four components of the current of the right sheet in the original frame

$$J^\mu = (J^0, J^x, J^y, J^z) = (0, 0, 0, K_o/\Delta)\,, \tag{20}$$

where Δ is the infinitesimal width of the sheets. J^0 is proportional to the surface charge density σ:

$$J^0 = \sigma c/\Delta\,, \tag{21}$$

and is zero in the original frame. Under boost we have

$$\underline{J}^\mu = L^\mu_{\ \nu} J^\nu\,. \tag{22}$$

This, together with the entries of the boost matrix

$$L^\mu_{\ \nu} = \begin{pmatrix} \gamma & & & -\gamma\beta \\ & 1 & & \\ & & 1 & \\ -\gamma\beta & & & \gamma \end{pmatrix}, \tag{23}$$

yields for the right sheet

$$\underline{\sigma} = -\,\gamma\beta K_o/c\,, \tag{24}$$

$$\underline{K}^z/c = \gamma K_{o/c}\,. \tag{25}$$

The left sheet has $J^z = -K_o/(c\Delta)$ and therefore the boosted charges and currents differ in sign

$$\underline{\sigma} = +\,\gamma\beta K_o/c\,, \tag{26}$$

$$\underline{K}^z/c = -\,\gamma K_{o/c}\,. \tag{27}$$

We can check our result by recognizing that the electric field in the y direction in the boosted frame that of a parallel plate capacitor with surface charges $+\underline{\sigma}$ and $-\underline{\sigma}$ on the left and right sheets:

$$\underline{E}^y = \underline{\sigma} = \gamma\beta K_o/c\,. \tag{28}$$

This agrees with the first part of (d). The magnetic field in the x direction is similarly

$$\underline{B}^x = \underline{K}^z/c = \gamma K_o/c\,, \tag{29}$$

and also agrees with the first part of (d).

(e) We will now compute the total energy in the boosted frame, $\boldsymbol{\beta} = \hat{\boldsymbol{y}}$. It is important to recognize that the mechanical stress tensor must also be boosted according to the general rule:

$$\underline{\Theta}^{\mu\nu} = L^{\mu}_{\ \rho} L^{\nu}_{\ \sigma} \Theta^{\rho\sigma} \tag{30}$$

(i) The energy density in the columns is

$$\underline{\Theta}^{00}_{\text{mech}} = \gamma^2 \Theta^{00}_{\text{mech}} + (-\gamma\beta)^2 \Theta^{yy}_{\text{mech}} \tag{31}$$

Integrating over the volume of the columns we find the total energy density. In this integration the separation between the sheets is length contracted $d \to d/\gamma$ yielding for the four columns

$$\int_V d^3\underline{\boldsymbol{r}}\, \underline{\Theta}^{00}_{\text{mech}} = A_{\text{col}} \frac{d}{\gamma} \left[4\gamma^2 \frac{M_{\text{col}} c^2}{d A_{\text{col}}} - 4\gamma^2\beta^2 \frac{\frac{1}{2}(K_o/c)^2 L^2}{4 A_{\text{col}}} \right] \tag{32}$$

$$= 4\gamma M_{col} c^2 - \gamma\beta^2 \left[L^2 d \tfrac{1}{2} (K_o/c)^2 \right] \tag{33}$$

(ii) The electromagnetic stress follows from the transformed fields:

$$\underline{B}^x = \gamma B^x = \gamma \frac{K_o}{c}\,, \tag{34}$$

$$\underline{E}^z = -\gamma\beta B^x = -\gamma\beta \frac{K_o}{c}\,. \tag{35}$$

So the electromagnetic energy density in between the sheets is

$$\underline{\Theta}^{00} = \tfrac{1}{2}(\underline{\boldsymbol{E}}^2 + \underline{\boldsymbol{B}}^2)\,, \tag{36}$$

$$= \tfrac{1}{2}(K_o/c)(\gamma^2\beta^2 + \gamma^2)\,, \tag{37}$$

and the total electromagnetic energy is therefore

$$\int_V d^3\underline{\boldsymbol{r}}\, \underline{\Theta}^{00}_{\text{em}} = \left[L^2 d \tfrac{1}{2} (K_o/c)^2 \right] (\gamma + \gamma\beta^2)\,. \tag{38}$$

(iii) Adding the two contributions, the terms proportional to $[L^2 d \frac{1}{2}(K_o/c)^2]\gamma\beta^2$ cancel, and we find

$$\int_V d^3\underline{\boldsymbol{r}}\, \underline{\Theta}^{00}_{\text{tot}} = \gamma \left(4 M_{\text{col}} c^2 + \left[L^2 d \tfrac{1}{2} (K_o/c)^2 \right] \right)\,. \tag{39}$$

This, as expected, is simply

$$\gamma M_{\text{tot}} c^2\,, \tag{40}$$

where $M_{\text{tot}} c^2$ was the rest energy computed in part (b).