Electromagnetism 1

Torque on a cylinder

The constitutive relation is a relation between the macroscopic electrical current density in a medium and the applied fields. Recall that for a normal isotropic conductor *at rest* in an electric ($\boldsymbol{E}$) and magnetic field ($\boldsymbol{B}$) the constitutive relation in a linear response approximation is known as Ohm's law: $\boldsymbol{J} = \sigma \boldsymbol{E}$.

a) (2 points) For most materials, a symmetry principle forbids a generalized Ohm's law in the rest frame of the material of the form:

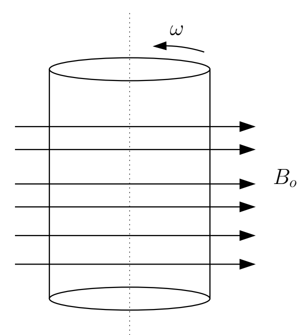
$$\boldsymbol{J} = \sigma \boldsymbol{E} + \sigma_B \boldsymbol{B} \,. \tag{1}$$

Explain.

b) (6 points) By making a Lorentz transformation for small velocities, deduce the familiar constitutive relation for a normal conductor *moving* non-relativistically with velocity $\boldsymbol{u}$ in an electric and magnetic field from the rest frame constitutive relation, Eq. ().

c) (4 points) Now consider a solid conducting cylinder of radius R and conductivity σ rotating rather slowly with constant angular velocity ω in a uniform magnetic field B_o perpendicular to the axis of the cylinder as shown below. Determine the current flowing in the cylinder.

d) (8 points) Determine the torque required to maintain the cylinder's constant angular velocity. Assume that the skin depth is much larger than the radius of the cylinder.



Solution

a) Parity forbids a constitutive relation including a magnetic field. Specifically, σ_B would have to be a pseudo-scalar, since $\boldsymbol{J}$ is a vector and $\boldsymbol{B}$ is a pseudo-vector. But, if the interactions of the medium are invariant under parity, and the ground state is parity symmetric, then medium expectation value of any pseudoscalar quantity is zero.

b) In a frame where the conductor is at rest

$$\underline{\boldsymbol{J}} = \sigma \underline{\boldsymbol{E}} \tag{2}$$

the charge density $\rho = 0$. Make a Lorentz transformation from the conductor's rest frame to the lab frame, *i.e.* a frame moving with velocity $-\boldsymbol{u}$ relative to the conductor, so that the lab observer sees the conductor moving with velcoity $\boldsymbol{u}$. We have

$$J^\mu = \Lambda^\mu_{\ \nu} \underline{J}^\nu \,. \tag{3}$$

Here the $\underline{J}$ are the currents in the conductor frame, J are the currents in the lab frame.

To first order in $\boldsymbol{u}$ the Lorentz transformation matrix is

$$\Lambda^\mu_{\ \nu} = \begin{pmatrix} \gamma & \gamma u \\ \gamma u & \gamma \end{pmatrix} \approx \begin{pmatrix} 1 & u \\ u & 1 \end{pmatrix} \tag{4}$$

Thus

$$\boldsymbol{J} \approx \boldsymbol{u} \underbrace{\underline{\rho}}_{=0} + \underline{\boldsymbol{J}} = \sigma \underline{\boldsymbol{E}} \tag{5}$$

We need to use the Lorentz transformation rule to relate $\underline{\boldsymbol{E}}$ to $\boldsymbol{E}$ and $\boldsymbol{B}$.

The transformation rules for the $\boldsymbol{E}$ and $\boldsymbol{B}$ fields are

$$E_\parallel = \underline{E}_\parallel \tag{6}$$
$$B_\parallel = \underline{B}_\parallel \tag{7}$$
$$E_\perp = \gamma \underline{E}_\perp - \gamma \boldsymbol{u}/c \times \underline{\boldsymbol{B}} \tag{8}$$
$$B_\perp = \gamma \underline{B}_\perp + \gamma \boldsymbol{u}/c \times \underline{\boldsymbol{E}} \tag{9}$$

and the inverse results

$$\underline{E}_{\parallel} = E_{\parallel} \tag{10}$$
$$\underline{B}_{\parallel} = B_{\parallel} \tag{11}$$
$$\underline{E}_{\perp} = \gamma E_{\perp} + \gamma \boldsymbol{u}/c \times \boldsymbol{B} \approx E_{\perp} + \boldsymbol{u}/c \times \boldsymbol{B} \tag{12}$$
$$\underline{B}_{\perp} = \gamma B_{\perp} - \gamma \boldsymbol{u}/c \times \boldsymbol{E} \tag{13}$$

So the constitutive relation becomes to first order

$$\boldsymbol{J} = \sigma(\boldsymbol{E} + \frac{\boldsymbol{u}}{c} \times \boldsymbol{B}) \tag{14}$$

Clearly the constitutive relation takes the form $\boldsymbol{J} = \sigma \boldsymbol{f}$ where $\boldsymbol{f}$ is the Lorentz force.

c) Using the result

$$\boldsymbol{J} = \sigma(\boldsymbol{u}/c \times \boldsymbol{B}_o)\,, \tag{15}$$

we find in cylindrical coordinates

$$\boldsymbol{J}(\rho, \phi) = -\sigma \frac{\omega \rho B_o}{c} \cos\phi \hat{\boldsymbol{z}}\,. \tag{16}$$

We see that the electrons (which carry negative charge) flow up the cylinder at $\phi = 0$ and down the cylinder at $\phi = \pi$.

d) The Lorentz force on the current induces a torque:

$$\tau = \int d^3\boldsymbol{r}\, \boldsymbol{r} \times \left(\frac{\boldsymbol{J}}{c} \times \boldsymbol{B}_o \right)\,, \tag{17}$$
$$= L \int \rho d\rho d\phi \left[\frac{\boldsymbol{J}}{c}(\boldsymbol{r} \cdot \boldsymbol{B}_o) - (\boldsymbol{r} \cdot \boldsymbol{J}/c)\boldsymbol{B}_o \right]\,, \tag{18}$$

where L is the length of the cylinder. The second term in square braces integrates to zero while the first terms gives

$$\tau = L \int_0^R \rho d\rho \int d\phi \left[(-\sigma \frac{\omega \rho B_o}{c^2} \cos\phi \hat{z})(\rho \cos\phi B_o) \right]\,, \tag{19}$$
$$= -L\hat{\boldsymbol{z}} \frac{\left(\pi \sigma \omega R^4 B_o^2\right)}{4c^2}\,. \tag{20}$$

This is the torque by the magnetic field on the cylinder. To maintain a constant angular velocity we need an external torque per unit length of

$$\frac{\tau}{L} = +\hat{\boldsymbol{z}} \frac{\left(\pi \sigma \omega R^4 B_o^2\right)}{4c^2}\,. \tag{21}$$

Notes:

- An alternative way to derive this is to equate the work done per time by the external torque, $\tau \cdot \omega$, with the energy dissipation

$$\tau \cdot \omega = \int d^3\boldsymbol{r} \frac{\boldsymbol{J} \cdot \boldsymbol{J}}{\sigma} \tag{22}$$

$$= L \frac{\sigma\omega^2}{4c^2} B_o^2 \pi R^4 \tag{23}$$

- We next evaluate this numerically for copper. Expressing the torque in terms of the skin depth (which is taken from Wikipedia):

$$\delta = \sqrt{\frac{2c^2}{\sigma\omega}} = 6.5\,\mathrm{cm}/\sqrt{f_{Hz}} \tag{24}$$

  We find

$$\frac{\tau}{L} = \frac{R^4 B_o^2}{\delta^2} \frac{\pi}{2} \tag{25}$$

  Converting to MKS and Tesla

$$B_o^2 \to \frac{B_o^2}{\mu_o} = 1\,\frac{J}{m^3}\,8 \times 10^5 \left(\frac{B_o}{\mathrm{Tesla}}\right)^2 \tag{26}$$

  So we find

$$\frac{\tau}{L} \approx 3\,\mathrm{N} \left(\frac{R}{\mathrm{cm}}\right)^4 \left(\frac{f}{Hz}\right) \left(\frac{B_o}{\mathrm{Tesla}}\right)^2 \qquad \text{with} \qquad R \ll \frac{6.5\,\mathrm{cm}}{\sqrt{f \text{ in Hz}}} \tag{27}$$

It is also interesting to calculate the current flowing through each hemi-cylinder of the wire.

$$\frac{I}{c} = \int \rho d\rho \int_{-\pi/2}^{\pi/2} d\phi \boldsymbol{J}(\rho, \phi)/c \tag{28}$$

$$= -\frac{2}{3}\sigma \frac{\omega R^3 B_o}{c^2} \hat{\boldsymbol{z}} \tag{29}$$

$$= -\frac{4}{3} \frac{R^3 B_o}{\delta^2} \tag{30}$$

Or in MKS

$$\frac{I}{c} \to \sqrt{\mu_o} I \tag{31}$$

$$\boldsymbol{B} \to \frac{\boldsymbol{B}}{\sqrt{\mu_o}} \tag{32}$$

which evaluates to a shockingly large current

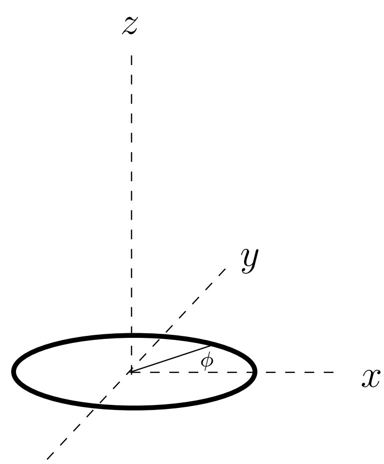
$$I = -\frac{4}{3}\frac{R^3 B_o}{\delta^2 \mu_o} \tag{33}$$

$$= 310\text{Amps}\left(\frac{f}{\text{Hz}}\right)\left(\frac{B}{\text{Tesla}}\right)\left(\frac{R}{\text{cm}}\right)^3 \tag{34}$$

Electromagnetism 2

Oscillating current on a ring

A current is driven through a ring of radius R in the xy plane (see below). Using complex notation, the current has a harmonic time dependence, $\boldsymbol{J}(t, \boldsymbol{r}) = e^{-i\omega t}\boldsymbol{J}(\boldsymbol{r})$, and the spatial dependence is $\boldsymbol{J}(\boldsymbol{r}) = I_0 \sin(\phi)\, \delta(\rho - R)\delta(z)\hat{\boldsymbol{\phi}}$.



a) (4 points) Sketch the current flow at time $t = 0$ and $t = \pi/\omega$, and determine the charge density $\rho(t, \boldsymbol{r})$. Show that it corresponds to an oscillating electric dipole, and determine the electric dipole moment.

b) In the long wavelength limit, and in the radiation zone, determine each of the following quantities in the xz plane at $y = 0$:

   (a) (6 points) The vector potential $\boldsymbol{A}(t, \boldsymbol{r})$ in the Lorentz gauge.

   (b) (4 points) The magnetic field $\boldsymbol{B}(t, \boldsymbol{r})$.

   (c) (4 points) The (time averaged) angular distribution of the radiated power, $dP/d\Omega$.

c) (2 points) What is the polarization of the radiated electric field when viewed along the z axis ?

Solution

We use Heavyside-Lorentz units.

a) Using current conservation, $\partial_t \rho + \nabla \cdot \boldsymbol{J} = 0$ and a harmonic time dependence, $\rho(t, \boldsymbol{r}) = e^{-i\omega t}\rho(\boldsymbol{r})$,

$$-i\omega\rho(\boldsymbol{r}) = -\nabla \cdot \boldsymbol{J}(\boldsymbol{r}) = -\frac{1}{R}\frac{\partial}{\partial\phi}J^{\phi}\,. \tag{1}$$

Thus

$$\rho(\boldsymbol{r}) = -\frac{I_o \cos\phi}{-i\omega R}\delta(z)\delta(\rho - R) \tag{2}$$

Note, the charge distribution gives rise to a net dipole moment

$$\boldsymbol{p} = \int d^3\boldsymbol{r}\rho(\boldsymbol{r})\boldsymbol{r} = \frac{I_o R}{-i\omega}(-\pi\hat{\boldsymbol{x}}) \tag{3}$$

pointed along the negative $\hat{\boldsymbol{x}}$ direction. If this is recognized then the remainder of this problem is just quoting the results of the electric dipole radiation.

b) a) In the dipole approximation we have

$$\begin{aligned} \boldsymbol{A}(t, \boldsymbol{r}) &= \frac{e^{-i\omega t + ikr}}{4\pi r}\int d^3\boldsymbol{r}'\boldsymbol{J}(\boldsymbol{r}')/c \\ &= \frac{e^{-i\omega t + ikr}}{4\pi r}\int \rho d\rho d\phi dz\, \hat{\boldsymbol{\phi}}\, (I_0/c)\sin\phi\, \delta(\rho - R)\delta(z)\,. \end{aligned} \tag{4}$$

With $\hat{\boldsymbol{\phi}} = -\sin\phi\hat{\boldsymbol{x}} + \cos\phi\hat{\boldsymbol{y}}$ we obtain

$$\boldsymbol{A}(t, \boldsymbol{r}) = \frac{e^{-i\omega t + ikr}}{4\pi r} R(I_0/c)\pi\,(-\hat{\boldsymbol{x}})\,, \tag{5}$$

$$= \frac{e^{-i\omega t + ikr}}{4\pi r}\frac{-i\omega}{c}\boldsymbol{p} \tag{6}$$

b) Then

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}\,, \tag{7}$$

$$= \boldsymbol{n} \times \frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{A}(t, \boldsymbol{r})\,, \tag{8}$$

$$= \frac{e^{-i\omega t + ikr}}{4\pi r}(\boldsymbol{n} \times -\hat{\boldsymbol{x}})(-ikR)(I_0/c) \tag{9}$$

$$= \frac{e^{-i\omega t + ikr}}{4\pi r}\cos\theta(-\hat{\boldsymbol{y}})(-ik\pi R)(I_0/c) \tag{10}$$

c) The radiated power is

$$\frac{dP}{d\Omega} = \frac{c}{2}\mathrm{Re}(r^2\boldsymbol{n}\cdot(\boldsymbol{E}\times\boldsymbol{B}^*))\,. \tag{11}$$

With $\boldsymbol{E} = -\boldsymbol{n}\times\boldsymbol{B}$, we have

$$\boldsymbol{n}\cdot(-\boldsymbol{n}\times\boldsymbol{B})\times\boldsymbol{B}^* = |\boldsymbol{B}|^2\,, \tag{12}$$

and

$$\begin{aligned} \frac{dP}{d\Omega} &= \frac{c}{2}r^2|\boldsymbol{B}|^2 && (13)\\ &= \frac{c}{32\pi^2}\cos^2\theta\,(\pi kRI_0/c)^2 && (14) \end{aligned}$$

It is perhaps useful to convert to MKS units:

$$\begin{aligned} \frac{I_0}{c} &\rightarrow \sqrt{\mu_o}I && (15)\\ c &\rightarrow \frac{1}{\sqrt{\mu_o\epsilon_o}} && (16) \end{aligned}$$

and using $\sqrt{\mu_o/\epsilon_o} = 376\,\mathrm{Ohm}$ we find

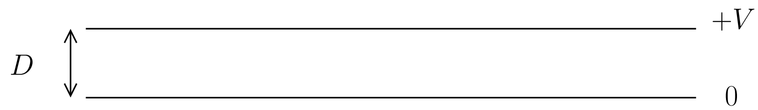
$$\frac{dP}{d\Omega} = 376\,\mathrm{Watts}\left(\frac{I_0}{\mathrm{amps}}\right)^2\frac{(kR)^2}{32}\cos^2\theta \tag{17}$$

c) Since the magnetic field is in the $-\hat{\boldsymbol{y}}$ direction, for light propagating along the z axis the electric field is in the $-\hat{\boldsymbol{x}}$ direction, *i.e.* along the direction of the dipole moment.

Electromagnetism 3

Parameters of an electron tube

Consider an idealized electron tube (diode) consisting of infinite planar cathode and anode separated by a distance D in the z direction (see below). The cathode (at $z = 0$) may be regarded as an infinite supply of free electrons at rest. The anode (at $z = D$) is at potential $+V$ relative to the cathode. (V is sufficiently small that Newtonian physics applies.) The device is evacuated, so that only electrons are between the two electrodes. The current through such a device is determined by the flow of the charge of these electrons from the cathode ($z = 0$) to the anode ($z = D$).



a) (10 points) Use Poisson's equation, the equation of continuity, and the conservation of energy to derive a differential equation for the electric potential $\Phi(z)$ in steady state. Make sure you have the sign correct, and state the boundary conditions explicitly.

b) (6 points) Find $\Phi(z)$ and use it to determine the current density J as a function of the parameters of the problem and physical constants. *Hint:* Try a scaling solution of the form $\Phi(z) \propto z^{\beta}$.

c) (4 points) Put in numbers for a centimeter-sized device and an anode potential of 300 volts to *estimate* the impedance typical of electron tube circuits.

Solution

a) Let $v(z)$ = speed of electrons at distance c from the cathode.

Total energy of electron $= mv(z)^2/2 - e\Phi(z) = 0$, so $v(z) = (2e\Phi(z)/m)^{1/2}$.

Continuity: Current density $J = v(z)\rho(z)$ is constant, independent of z.

$$\begin{aligned}\text{Poisson: } \nabla^2\Phi &= -\frac{\rho}{\epsilon_0} && (1)\\ \frac{d^2\Phi}{dz^2} &= +\frac{|J|}{\epsilon_0 v(z)} = \left[\frac{|J|}{\epsilon_0}\sqrt{\frac{m}{2e}}\right]\Phi(z)^{-1/2}. && (2)\end{aligned}$$

Boundary conditions are $\Phi(0) = 0, \Phi(D) = V$. Note that there is no boundary condition on $\frac{d\Phi}{dz}$ at $z = 0$.

b) Hypothesize a solution of the form $\Phi(z) = Az^\beta$.

$$\frac{d^2\Phi}{dz^2} = A\beta(\beta-1)z^{\beta-2} = KA^{-1/2}\beta^{-1/2}. \tag{3}$$

(Here K is the factor in square brackets in equation 2.)

This works if $\beta - 2 = -1/2$, *i.e.*, $\beta = 4/3$ and $\frac{4}{9}A^{3/2} = K$. The solution is

$$\Phi(z) = \left(\frac{9J}{4\epsilon_0}\right)^{2/3}\left(\frac{m}{2e}\right)^{1/3} z^{4/3}. \tag{4}$$

The boundary condition that $\Phi(D) = V$ leads to

$$|J| = \frac{4\epsilon_0}{9}\sqrt{\frac{2e}{m}}\,\frac{V^{3/2}}{D^2}. \tag{5}$$

c) To substitute units, insert a factor of $c = 1/\sqrt{\mu_o\epsilon_o}$, note that

$$m_e c^2 = 0.5\,\text{MeV} = 0.5\times 10^6\,\text{eV}\,,$$

and recall that $\sqrt{\mu_o/\epsilon_o} = 376\,\Omega$. We find (using ($e\cdot V = 300\,\text{eV}$)

$$\begin{aligned}|J| &= \frac{4}{9}\left(\frac{V}{376\,\Omega}\right)\frac{1}{D^2}\sqrt{\frac{2e\cdot V}{mc^2}} && (6)\\ &= 121\,\frac{\text{Amps}}{\text{meter}^2}\left(\frac{V}{300\,\text{Volts}}\right)^{3/2}\left(\frac{\text{cm}}{D}\right)^2 && (7)\end{aligned}$$

So taking the plate area to be $1\,\text{cm}^2$

$$\begin{aligned}\frac{V}{I} &= 1.59\times 376\,\Omega\sqrt{\frac{mc^2}{e\cdot V}}\,\frac{D^2}{\text{Area}} && (8)\\ &= 25000\,\Omega\sqrt{\frac{300\,\text{Volts}}{V}}\left(\frac{D^2/\text{Area}}{1}\right) && (9)\end{aligned}$$