Vector and scalar potential

$\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow \exists\, \vec{A}$ such that $\vec{B} = \vec{\nabla}\times\vec{A}$

$$\Rightarrow \vec{\nabla}\times\vec{E} + \frac{1}{c}\partial_t \vec{\nabla}\times\vec{A} = 0$$

$$\Rightarrow \vec{\nabla}\times\left(\vec{E} + \frac{1}{c}\partial_t\vec{A}\right) = 0$$

$\Rightarrow \exists$ scalar potential ϕ

$$\vec{E} + \frac{1}{c}\partial_t\vec{A} = -\vec{\nabla}\phi$$

$$\Rightarrow \vec{B} = \vec{\nabla}\times\vec{A}$$
$$\vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\partial_t\vec{A}$$

gauge invariant $\quad \vec{A} \to \vec{A} + \vec{\nabla}\chi$

$$\phi \to \phi - \frac{1}{c}\partial_t\chi$$

Inhomogeneous Maxwell equations in terms of gauge potential

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho \Rightarrow -\vec{\nabla}^2\phi - \frac{1}{c}\vec{\nabla}\partial_t\vec{A} = 4\pi\rho$$

$$\vec{\nabla}\times\vec{B} - \frac{1}{c}\partial_t\vec{E} = \frac{4\pi}{c}\vec{j} \Rightarrow$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) + \frac{1}{c}\partial_t\vec{\nabla}\phi + \frac{1}{c^2}\partial_t^2\vec{A} = \frac{4\pi}{c}\vec{j}$$

We can simplify these equations by a suitable gauge choice

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \vec{\nabla}^2 A$$

<u>Coulomb gauge</u> $\vec{\nabla}\cdot\vec{A}=0$

$$\Rightarrow \quad -\vec{\nabla}^2\vec{A}+\frac{1}{c^2}\partial_t^2\vec{A}=\frac{4\pi}{c}\vec{j}-\frac{1}{c}\partial_t\vec{\nabla}\phi$$

$$\vec{\nabla}^2\phi=-4\pi\rho$$

instantaneous Coulomb potential

The rhs is transverse:

$$\vec{\nabla}\cdot\left(\frac{4\pi}{c}\vec{j}-\frac{1}{c}\partial_t\vec{\nabla}\phi\right)$$

$$=\frac{4\pi}{c}\vec{\nabla}\cdot\vec{j}-\frac{1}{c}\partial_t\vec{\nabla}^2\phi=\frac{4\pi}{c}(\underbrace{\vec{\nabla}\cdot\vec{j}+\partial_t\rho}_{=0})$$

by continuity eq.

longitudinal component of $\vec{j}$ is cancelled by $\frac{1}{c}\partial_t\phi$

It is always possible to satisfy the Coulomb gauge $\vec{\nabla}\cdot\vec{A}'=\vec{\nabla}A+\vec{\nabla}^2\chi=0$

just solve for χ

<u>Lorentz gauge</u> $\vec{\nabla}\cdot\vec{A}+\frac{1}{c}\partial_t\phi=0$

$$\Rightarrow \quad -\vec{\nabla}^2A+\frac{1}{c^2}\partial_t^2A=\frac{4\pi}{c}j$$

$$-\vec{\nabla}^2\phi+\frac{1}{c^2}\partial_t^2\phi=4\pi\rho$$

We have no instantenous Coulomb potential because of the ∂_t^2 derivatives

Can we always satisfy the Lorentz gauge

$$\vec{\nabla} A' + \frac{1}{c}\partial_t \phi' = \vec{\nabla}\vec{A} + \vec{\nabla}^2 \chi + \frac{1}{c}\partial_t \phi - \frac{1}{c^2}\partial_t^2 \chi = 0$$

$$\Rightarrow \vec{\nabla}\cdot\vec{A} + \frac{1}{c}\partial_t \phi = -\vec{\nabla}^2 \chi + \frac{1}{c^2}\partial_t^2 \chi$$

can be solved for χ

still $\exists$ residual gauge invariant for functions that satisfy

$$-\vec{\nabla}^2 \chi + \frac{1}{c^2}\partial_t \chi = 0$$

$$\Box = \vec{\nabla}^2 - \frac{1}{c^2}\partial_t^2$$

Green's functions

$$\Box \psi = -4\pi f \quad (**)$$

Green function $\quad \Box G = -4\pi \delta^3(\vec{r}-\vec{r}')\times\delta(t-t')$

$$G = G(\vec{r}, \vec{r}', t, t')$$

Then the solution of $(**)$ is given by

$$\psi = \int d^3r' dt' \, G(\vec{r}, \vec{r}', t, t') f(\vec{r}', t') + \chi$$

with $\Box \chi = 0$

χ can be fixed by boundary conditions.

We look for a translational invariant (11)
solution. So we can put one of the
arguments to 0 $\vec{r}' \to 0$, $t' \to 0$

$$\Rightarrow \Box G = -4\pi\, \delta^3(\vec{r})\, \delta(t)$$

Fourier transform

$$G(r,t) = \int d^3k\, d\omega\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}\, G(\vec{k},\omega)$$

$$\delta^3(\vec{r})\,\delta(t) = \int \frac{d^3k\, d\omega}{(2\pi)^4}\, e^{i(\vec{k}\vec{r}-\omega t)}$$

$$\Rightarrow \left(-k^2 + \frac{\omega^2}{c^2}\right) G(k,\omega) = \frac{-4\pi}{(2\pi)^4}$$

$$\Rightarrow G(\vec{k},\omega) = -\frac{1}{4\pi^3}\, \frac{1}{\frac{\omega^2}{c^2} - k^2}$$

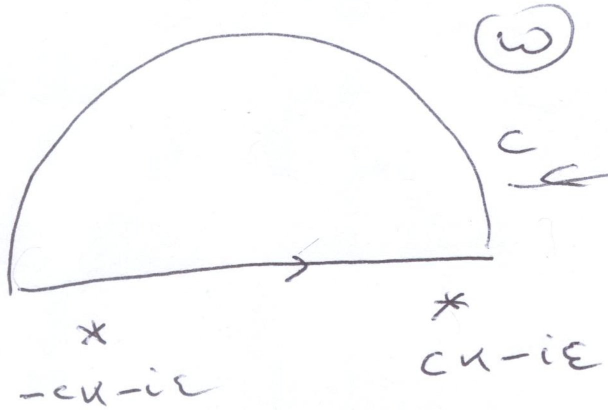
We implement the boundary condition
by an $i\varepsilon$ presription $\omega \to \omega \pm i\varepsilon$

<u>Returded and advanced Green's function</u>

$$G(\vec{r},t) = \int d^3k\, e^{ikr} \int_{-\infty}^{\infty} \frac{e^{-i\omega t}}{\frac{\omega^2}{c^2} - k^2}\, \frac{(-1)}{(4\pi)^3}$$

$$\frac{\omega^2}{c^2} - k^2 \longrightarrow \frac{(\omega+i\varepsilon)^2}{c^2} - k^2 \Rightarrow \text{poles}$$

are at $\omega = \pm ck - i\varepsilon$



$\omega + i\varepsilon$

$e^{-i\omega t} \to 0$ for $t < 0$

$$\Rightarrow \int_{-\infty}^{\infty} d\omega \ldots = \int_C d\omega \ldots = 0$$

$$\Rightarrow \text{for } t<0 \quad \int_{-\infty}^{\infty} \frac{e^{-i\omega t}}{\frac{\omega^2}{\kappa^2} - k^2} = 0$$

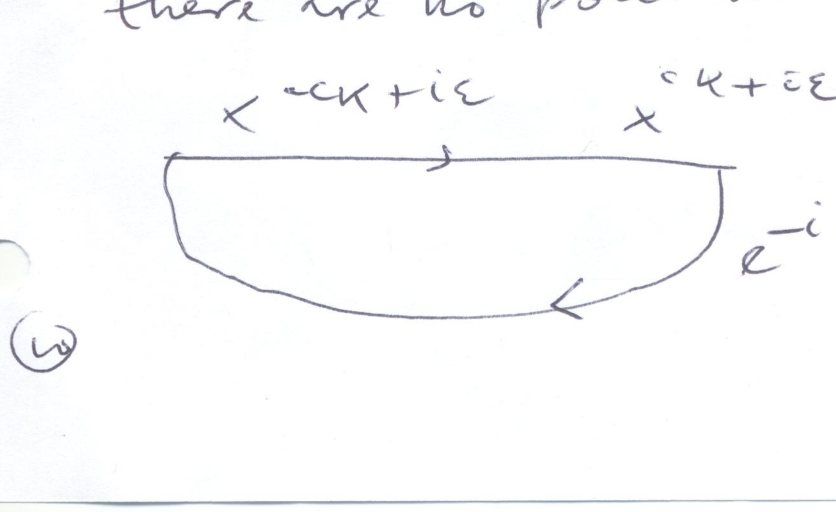
$$\Rightarrow G(\vec{r}, t) = 0 \quad \text{for } t < 0$$

This is the retarded Green's function

The advanced Green's function is defined by

$$G(\vec{r}, t) = 0 \quad \text{for } t > 0$$

This is achieved by $\omega \to \omega - i\varepsilon$ so that there are no poles in the lower half plane



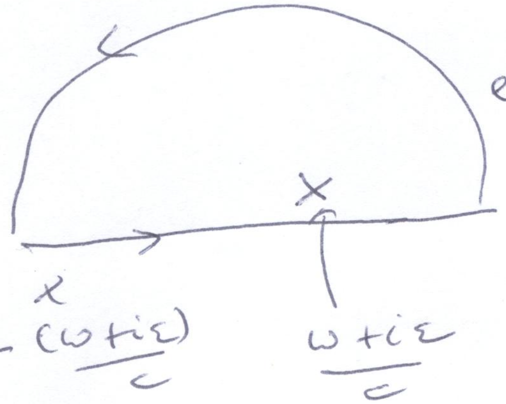
$e^{-i\omega t} \to 0$ for $t > 0$

Calculation of $G(\vec{r},t)$

$$G_R(\vec{r},t) = \int d\omega\, e^{-i\omega t} \frac{1}{4\pi^3} \int 2\pi k^2 dk \sin\theta\, d\theta$$

$$\times \frac{e^{ikr\cos\theta}}{k^2 - \frac{(\omega+i\varepsilon)^2}{c^2}}$$

k2



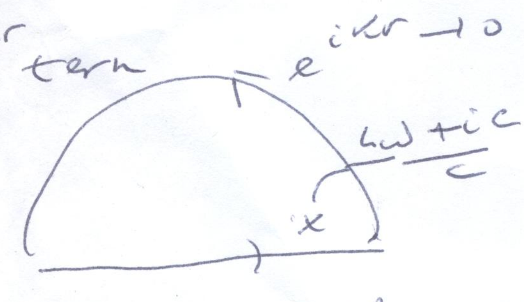
- first do θ integral $\cos\theta = x$

$$\int_0^\pi \sin\theta\, d\theta\, e^{ikr\cos\theta} = \frac{1}{ikr}\left(e^{ikr} - e^{-ikr}\right)$$

$$\int_0^\infty \frac{k^2 dk \left(e^{ikr} - e^{-ikr}\right)}{\left(k^2 - \frac{(\omega+i\varepsilon)^2}{c^2}\right) ikr} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{k^2 dk \left(e^{ikr} - e^{-ikr}\right)}{\left(k^2 - \frac{(\omega+i\varepsilon)^2}{c^2}\right) ikr}$$

$$\frac{1}{\left(k - \frac{\omega+i\varepsilon}{c}\right)\left(k + \frac{\omega+i\varepsilon}{c}\right)}$$

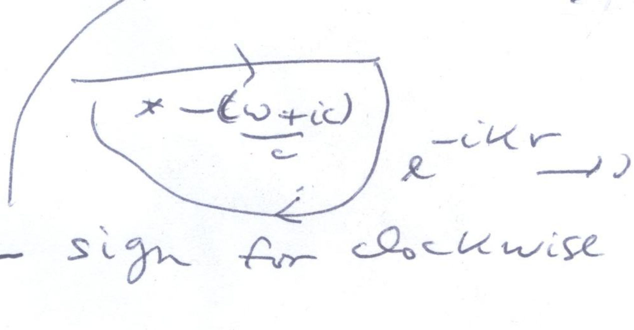
e^{ikr} term



$$\frac{1}{2}\, 2\pi i\, \frac{\left(\frac{\omega+i\varepsilon}{c}\right)^2}{2\left(\frac{\omega+i\varepsilon}{c}\right)^2}\, e^{i\left(\frac{\omega+i\varepsilon}{c}\right) r}$$

$$-\frac{1}{2}\, \frac{2\pi i \left(\frac{\omega+i\varepsilon}{c}\right)^2 (-)\, e^{ik\frac{(\omega+i\varepsilon)}{c} r}}{-2\frac{(\omega+i\varepsilon)}{c}\left(-\frac{(\omega+i\varepsilon)}{c}\right)}$$

e^{-ikr} term



- sign for clockwise contour

The two integral are the same.

$$G_R(\vec{r},t) = \int d\omega\, e^{-i\omega t} \frac{2\pi}{4\pi^3} \frac{2\pi i 2}{4} e^{i\left(\frac{\omega \mp i\epsilon}{c}\right) r}$$

$$= \frac{1}{2\pi r} \int d\omega\, e^{\left(-i\omega t + i\frac{\omega r}{c}\right)}$$

$$= \frac{1}{2\pi r} 2\pi\, \delta\left(\frac{r}{c} - t\right)$$

$$= \frac{c}{r}\, \delta(r - ct)\, \theta(t)$$

because $r > 0$

For $G_A(r,t)$ the analogous calculation gives

$$G_A(\vec{r},t) = \frac{c}{r}\, \delta(r + ct)\, \theta(-t)$$

Therefore the solution of the PDE is given by

$$\psi(r,t) = \int d^3r'\, dt' \frac{\delta\left(t - t' - \frac{|\vec{r} - \vec{r}'|}{c}\right)}{|r - r'|} F(\vec{r}', t)$$

retarded time $\quad t_R = t - \dfrac{|\vec{r} - \vec{r}'|}{c}$

this is the time at which a signal must be emitted to be observed at $(\vec{r}, t)$

Solution for $\vec{A}$ and ϕ

$$\vec{A} = \int \frac{d^3r'}{|\vec{r} - \vec{r}'|} \vec{J}(\vec{r}', t_R)$$

$$\phi = \int \frac{d^3r'}{|\vec{r} - r'|} \rho(r', t_R)$$

no instanteous fields!

Radiation

source

$$\vec{j}(\vec{r},t) = \vec{j}_0(\vec{r})\, e^{-i\omega t}$$

$$\rho(\vec{r},t) = \rho_0(\vec{r})\, e^{-i\omega t}$$

Continuity eq. $\vec{\nabla}\, \vec{j}_0 = i\omega\rho_0$

Far away, the energy density per unit solid angle is constant. This is the radiation zone.

$$d\vec{s}\cdot\vec{P} = \frac{c}{4\pi}\,\vec{E}\times\vec{B}\cdot d\vec{s}$$

$d\vec{s}$

$$= \frac{c}{4\pi}(\vec{E}\times\vec{B})\cdot\hat{r}\; r^2 d\Omega$$

$$= \frac{c}{4\pi}(\vec{E}\times\vec{B})$$

this can only be constant if $\vec{E}\sim\frac{1}{r}$, $\vec{B}\sim\frac{1}{r}$

$$\vec{A} = \frac{1}{c}\int\frac{d^3r'}{|\vec{r}-\vec{r}'|}\,\vec{j}_0(\vec{r})\, e^{-i\omega t_R}$$

$$\phi = \int\frac{d^3r'}{|r-r'|}\,\rho_0(\vec{r})\, e^{-i\omega t_R}$$

we only need leading order terms in $\frac{1}{r}$

$$t_R = t - \frac{|r-r'|}{c} = t - \frac{1}{c}\sqrt{\vec{r}^2+\vec{r}'^2-2\vec{r}\cdot\vec{r}'}$$

$$= t - \frac{r}{c} - \frac{\hat{r}\cdot\vec{r}'}{c} + O\left(\frac{r'^2}{r^2}\frac{r}{c}\right)$$

$$\frac{1}{|r-r'|} = \frac{1}{r} + O\left(\frac{r'}{r^2}\right)$$

For $r \gg r'$ we can neglect the corrections

$$\Rightarrow A(\vec{r},t) = \frac{e^{-i\omega(t-\frac{r}{c})}}{rc} \int d^3r' \, e^{-i\omega\frac{\vec{r}\cdot\vec{r}'}{cr}} \vec{j}_0(r')$$

$$\phi(\vec{r},t) = \frac{e^{-i\omega(t-\frac{r}{c})}}{r} \int d^3r' \, e^{-i\omega\frac{\vec{r}\cdot\vec{r}'}{cr}} \rho_0(\vec{r}')$$

<u>Physical fields</u>

$$\vec{E} = -\frac{1}{c}\partial_t \vec{A} - \vec{\nabla}\phi$$

$$= \frac{e^{-i\omega(t-\frac{r}{c})}}{r} \int d^3r' \, e^{-i\omega\frac{(\vec{r}\cdot\vec{r}'}{cr}} \times \left(\frac{i\omega}{c^2}\vec{j}_0 - \frac{i\omega}{c}\frac{\vec{r}}{r}\rho_0\right) + O\left(\frac{1}{r^2}\right)$$

$$\vec{B} = \vec{\nabla}\times\vec{A} = \frac{i\omega}{c^2}\frac{\vec{r}}{r^2} \times e^{-i\omega(t-\frac{r}{c})} \times \int d^3r' \, e^{-i\omega\frac{\vec{r}\cdot\vec{r}'}{cr}} \vec{j}_0(r')$$

For a wave with $\vec{k} = \frac{\omega}{c}\frac{\vec{r}}{r}$ we have

$\vec{E} \perp k$:

$$\int d^3r' \, \frac{i\omega}{c}\rho_0 \, e^{-i\omega\frac{\vec{r}\cdot\vec{r}'}{cr}} \overset{\text{continuity eq.}}{=} \frac{1}{c}\int d^3r' \, \nabla' \vec{j}_0(r') \, e^{-i\omega\frac{\vec{r}\cdot\vec{r}'}{cr}}$$

$$= -\int d^3r \, \vec{j}_0(r') \left(-\frac{i\omega}{c^2}\frac{\vec{r}}{r}\right) e^{-i\omega\frac{\vec{r}\cdot\vec{r}'}{cr}}$$

$$\Rightarrow \vec{E}\cdot\frac{\vec{r}}{r} = 0 \Rightarrow \vec{E}\cdot\vec{k} = 0$$

- Longitudinal part of $\vec{j}_0$ is cancelled by $\frac{c\vec{r}}{r}\rho_0$.
- Longitudinal part of $\vec{j}_0$ does not contribute to $\vec{B}$

Because $-\frac{i\omega}{c}\frac{\vec{r}}{r}\rho_0$ is cancelled we have

$$\cancel{\vec{B} = \frac{\vec{k}}{c} \times \vec{E}} \qquad \vec{B} = \hat{k} \times \vec{E}$$

Radiated energy

$$\frac{dW}{dt\,dA} = \vec{P}\cdot\hat{r} \Rightarrow \frac{dW}{dt\,d\Omega} = r^2\,\vec{P}\cdot\hat{r}$$

$$= r^2 \frac{c}{8\pi} \mathrm{Re}\; \vec{E}^* \times \vec{B}\cdot\hat{r}$$

$$\vec{B} = \hat{k} \times \vec{E}$$

$$\vec{E}^* \times (\hat{k} \times \vec{E}) = \hat{k}\;\vec{E}^*\cdot E$$

$$\Rightarrow \frac{dW}{dt\,d\Omega} = \frac{cr^2}{8\pi}\cancel{k}\;\vec{E}^*\cdot E$$

$$= \frac{cr^2}{8\pi}\cancel{k}\cdot\frac{1}{r^2}\left|\int d^3r'\, e^{-i\omega\frac{\vec{r}\cdot\vec{r}'}{cr}}\, j_{0T}\frac{i\omega}{c^2}\right|^2$$

$$\vec{j}_T = \vec{j} - c\frac{\vec{r}}{r}\rho$$

$$\Rightarrow \frac{dW}{dt\,d\Omega} = \frac{c\omega^2}{c^3\,8\pi}\left|\int d^3r'\, e^{-i\vec{k}\cdot r'}\, j_T(r')\right|^2$$

Radiation by slowly moving charges

$$\vec{j} = q\, \vec{v}(t)\, \delta(\vec{r} - r_p(t))$$

$$\vec{A} = \frac{1}{c}\int d\vec{r}'\, dt'\, c\, \frac{\delta(|r - r'| - (t-t')c)}{|r - r'|}\, \vec{j}(\vec{r}', t')$$

$$= \frac{1}{c}\int dt'\, q\, \vec{v}(t') \cdot \frac{\delta(|\vec{r} - \vec{r}_p(t')| - (t-t')c)}{|\vec{r} - \vec{r}_p|}$$

$$\delta(f(x)) = \frac{1}{f'(x)}\, \delta(x)$$

$$\partial_t\, [\,|\vec{r} - \vec{r}_p(t')| - (t-t')\, c\,]$$

$$= c - \frac{(\vec{r} - \vec{r}_p) \cdot \dot{r}_p}{|\vec{r} - r_p|}$$

For $r \to \infty \quad \simeq c - \hat{r} \cdot \dot{\vec{r}}_p$

$$\Rightarrow A = \frac{q}{c}\, \frac{v(t_R)}{r}\, \frac{1}{1 - \frac{\hat{r} \cdot \dot{r}_p(t_R)}{c}} + O\left(\frac{1}{r^2}\right)$$

$$\phi = \int d^3r'\, dt'\, \frac{\delta(|r - r'| - (t-t')c)}{|r - r'|}\, \rho(r', t')$$

$$q\, \delta(\vec{r} - \vec{r}_p(t'))$$

$$\Rightarrow \phi = \frac{q}{r}\, \frac{1}{1 - \frac{\hat{r} \cdot \dot{\vec{r}}_p(t_R)}{c}} + O\left(\frac{1}{r^2}\right)$$

$$t_R \simeq t - \frac{r}{c}$$

$$\vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\partial_t \vec{A}$$

$$= -\frac{q}{r}\frac{-1}{\left(1-\frac{\hat{r}\cdot\dot{\vec{r}}_p}{c}\right)c}\left(\frac{\ddot{\vec{r}}_p\cdot\hat{r}}{c^2}\right)\hat{r}$$

$$-\frac{q}{c}\frac{\dot{v}(t_R)}{c} + O\left(\frac{v}{c}\right)$$

$$-\vec{\nabla}(\dot{\vec{r}}_p(t_R)) = \vec{\nabla}\left(\dot{r}_p\left(t-\frac{r}{c}\right)\right)$$

$$= -\ddot{\vec{r}}_p(t_R)\cdot\hat{r}\left(-\frac{1}{c}\right)\frac{\vec{r}}{r}$$

$$\vec{E} = -\frac{q}{c^2 r}\left(\vec{a} - (\vec{a}\cdot\hat{r})\hat{r}\right) + O\left(\frac{v}{c}\right) + O\left(\frac{1}{r^2}\right)$$

$$\vec{B} = (\vec{\nabla}\times\vec{A})_i = \frac{q}{c}\vec{\nabla}\times\left(\frac{v_p\left(t-\frac{r}{c}\right)}{r}\frac{1}{1-\frac{\hat{r}\,\dot{r}_p(t_R)}{c}}\right)$$

$$\uparrow$$

$$O\left(\frac{v}{c}\right)$$

$$= \frac{q}{c}\varepsilon_{ijk}\partial_j v_{p,k}\left(t-\frac{r}{c}\right)$$

$$= \frac{q}{c}\dot{v}_{p,k}\,-\frac{1}{c}\hat{r}_j\,\varepsilon_{ijk}$$

$$= -\frac{q}{c^2}\hat{r}\times\dot{v}_p \Rightarrow \vec{B} = \hat{r}\times\vec{E}$$

note that $\hat{r}\times\hat{r} = 0$

$$\vec{E}\cdot\hat{r} = -\frac{q}{c^2 r}(\vec{a}\cdot\hat{r} - \vec{a}\cdot\hat{r}) = 0$$

$\Rightarrow$ we have a transverse wave

Poynting vector

$$\vec{P} = \frac{c}{4\pi}(\vec{E} \times \vec{B}) = \frac{c}{4\pi}(\vec{E} \times (\hat{r} \times \vec{E})$$

$$= \frac{c}{4\pi} \vec{E}^2 \hat{r}$$

$$= \frac{c}{4\pi} \frac{q^2}{c^4 r^2} \left| \vec{a} - (\vec{a}\cdot\hat{r})\hat{r} \right|^2 \hat{r}$$

radiation per unit of solid angle.

$$\hat{r}\cdot d\vec{p} = r^2 d\Omega\, \vec{p}\cdot\hat{r}$$

$$= r^2 \frac{c}{4\pi} \frac{q^2}{c^4 r^2} \left| \vec{a}^2 - \underbrace{(a\cdot\hat{r})}_{= a\cos\theta}{}^2 \right| d\Omega$$

$$\Rightarrow \frac{dp}{d\Omega} = \frac{q^2 a^2}{4\pi c^3} \sin^2\theta$$

Total power $p = \int d\Omega \, dp/d\Omega =$

$$= \int d\varphi \sin\theta \, d\theta \, \frac{q^2 a^2}{4\pi c^3} \sin^2\theta$$

$$= 2\pi \frac{q^2 a^2}{4\pi c^3} \int \sin^3\theta \, d\theta$$

$$= \frac{2}{3} \frac{q^2 a^2}{c^3}$$ Larmor formula