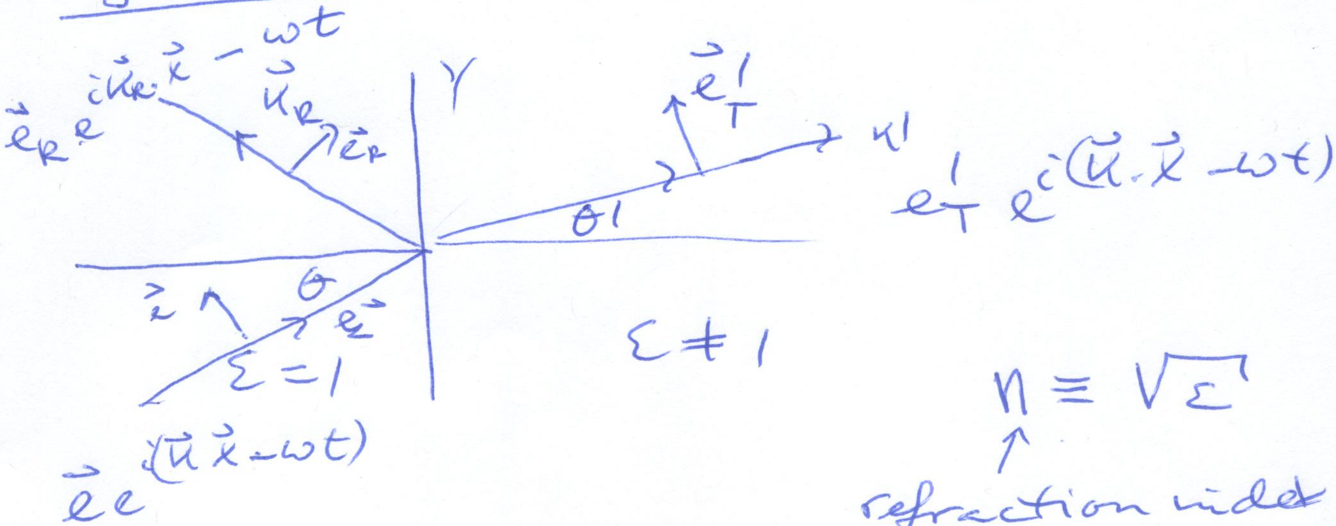
Reflection and Refraction



$n \equiv \sqrt{\varepsilon}$

refraction index

The frequencies of both sides are the same

$$\vec{k}^2 = \vec{k}_R^2 = \frac{\omega^2}{c^2} = \frac{\vec{k}'^2}{\varepsilon}$$

$\vec{E}$ has only component in the xy plane

$$\Rightarrow k_x^2 + k_y^2 = \frac{(k_x'^2 + k_y'^2)}{\varepsilon}$$

boundary conditions E_{tan} continuous

B_{tan} continuous

They must hold at $x = 0$ in the full ~~y~~ z plane $\Rightarrow$ k_y and k_z must be continuous. There is no k_z by assumption

So we get $\frac{\sin\theta'}{\sin\theta} = \frac{\cancel{k_y}}{k'}\frac{k}{\cancel{k_y}} = \frac{k}{k'} = \frac{1}{\sqrt{\varepsilon}}$

$\Rightarrow \quad n \sin\theta' = \sin\theta$

This is <u>Snel's law</u>

Next we determine the full solution.

$x > 0 \qquad \vec{E} = \vec{e}_T \, e^{i(k'_x x + k_y y - \omega t)}$

$\qquad B = \frac{c\vec{k}'}{\omega} \times \vec{E}$

$x < 0 \qquad \vec{E} = \vec{e}\, e^{ik_x x + k_y y - \omega t}$

$\qquad\qquad - \vec{e}_R \, e^{i(-k_x x + k_y y - \omega t)}$

$\qquad B = \frac{c}{\omega}\vec{k} \times \vec{E}$ $\qquad$ left moving wave

note that everywhere we have the same y-dependence.

$\vec{e} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \qquad \vec{e}_R = R\begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}$

$\qquad \vec{e}_T = T\begin{pmatrix} -\sin\theta' \\ \cos\theta' \end{pmatrix}$

continuity of the tangential (y) component gives

$\cos\theta(1-R) = T\cos\theta'$

(84)

For the magnetic field which has only a $\hat{z}$ component we obtain

$$\vec{B}_z = \frac{c}{\omega}(\vec{k} \times \vec{E})_z$$

$$\vec{k} \perp \vec{E} \quad = \frac{c}{\omega}\hat{z}\,|k|\,(1+R) \quad \frac{c}{\omega}(k_x \cos\theta - k_y(-\sin\theta))$$

$$\vec{k}_R = (-k_x, k_y)$$

$$\vec{e}_R = R(\sin\theta, \cos\theta)$$

$$\vec{B}_z^R = \frac{c}{\omega}(k_{Rx} e_{Ry} - k_{Ry} e_{Rx})$$

$$= \frac{c}{\omega}R(-k_x \cos\theta - k_y \sin\theta)$$

$$= -\frac{cR}{\omega}(\vec{k} \times \vec{e}) \qquad (\vec{e}_R \text{ was defined with a minus sign})$$

for B_z of transmitted wave

$$B_z^t = \hat{z}\,\frac{c k'}{\omega} T$$

$$\Rightarrow \frac{ck}{\omega}(1+R) = \frac{ck'}{\omega} T$$

$$\Rightarrow k(1+R) = k' T$$

$$\cos\theta\,(1-R) = T\cos\theta'$$

$$\Rightarrow 2 = T\left(\frac{k'}{k} + \frac{\cos\theta'}{\cos\theta}\right)$$

$$\Rightarrow T = \frac{2k\cos\theta}{k'\cos\theta + k\cos\theta'}$$

$$R = \frac{k'}{k} T - 1$$

$$= \frac{k' 2\cos\theta}{k'\cos\theta + k\cos\theta'} - 1$$

$$= \frac{k'\cos\theta - k\cos\theta'}{k'\cos\theta + k\cos\theta'}$$

$R + T = 1$ for $\theta = \theta' = 0$

Special cases

very good conductor $\quad \epsilon(\omega) = \epsilon + \frac{4\pi i\sigma}{\omega}$

$\sigma \to \infty \Rightarrow |\epsilon| \gg 1$

$\Rightarrow k' \gg k$

$\Rightarrow R \to 1$

So we see why metals are shiny.

Brewster angle $\quad R = 0$ if $k'\cos\theta = k\cos\theta'$

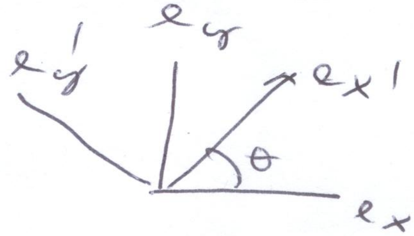
$$\Rightarrow \cos\theta = \frac{\sin\theta'}{\sin\theta}\cos\theta'$$

$$\Rightarrow \sin 2\theta = \sin 2\theta' \Rightarrow \theta' = \theta + \frac{\pi}{2}$$

$$\frac{1}{\sqrt{\epsilon}} = \frac{\sin\theta'}{\sin\theta} = \frac{\sin(\theta + \frac{\pi}{2})}{\sin\theta} = \frac{\cos\theta}{\sin\theta} \Rightarrow \tan\theta = \sqrt{\epsilon}$$

Helicity

$$e_{\pm} = \frac{\hat{e}_x \pm i \hat{e}_y}{\sqrt{2}}$$



$$e'_{\pm} = \frac{\hat{e}_x' \pm i \hat{e}_y'}{\sqrt{2}}$$

$$\begin{pmatrix} e_x' \\ e_y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} e_x \\ e_y \end{pmatrix}$$

$$\Rightarrow e'_{\pm} = \frac{(\cos\theta\, \hat{e}_x + \sin\theta\, \hat{e}_y)}{\sqrt{2}} \pm i \frac{(-\sin\theta\, \hat{e}_x + \cos\theta\, \hat{e}_y)}{\sqrt{2}}$$

$$= (e^{\mp i\theta} \hat{e}_x \pm i \hat{e}_y e^{\mp i\theta}) \frac{1}{\sqrt{2}}$$

$$= e^{\mp i\theta} \vec{e}_{\pm}$$

So $\vec{e}_{\pm}$ have helicity $\pm$ by definition of helicity.

Imaginary ε

Good conductor $\quad \varepsilon = \frac{4\pi i \sigma}{\omega}$

$k_y' = k_y$

Frequency is continuous

$$k_x'^2 + k_y'^2 = (k_x^2 + k_y^2)\,\varepsilon = \frac{\omega^2 \varepsilon}{c^2}$$

$\varepsilon \gg 1$

$k_y = k_y' \quad \Rightarrow \quad k_x' \gg k_x$

$$\Rightarrow k_x' \simeq \frac{\omega}{c}\sqrt{\varepsilon}$$

$$= \frac{\omega}{c}\sqrt{\frac{4\pi\sigma}{\omega}}\,\sqrt{i}$$

$$\Rightarrow E_T = \vec{e}_T\, e^{i k_y y + i x \frac{\omega}{c}\sqrt{\frac{2\pi\sigma}{\omega}}(1+i)}$$

$$\sqrt{i} = \frac{1+i}{\sqrt{2}}$$

damping factor $\quad e^{-x k \sqrt{\frac{2\pi\sigma}{\omega}}}$

penetration depth $\quad \dfrac{1}{k\sqrt{\frac{2\pi\sigma}{\omega}}}$

Continuity of the normal component of $\vec{D}$

$\vec{D}_n$ continuous

$x < 0 \qquad D_n = -\sin\theta + R\sin\theta$

$x > 0 \qquad D_n = -\varepsilon T \sin\theta'$

$\varepsilon = \frac{k'^2}{k^2} \Rightarrow (1+R)\sin\theta = \frac{k'^2}{k^2} T \sin\theta'$

<u>Snel</u> $\quad \frac{k'}{k}\sin\theta' = \sin\theta$

$\Rightarrow$

$(1+R)\sin\theta = \frac{k'}{k} T \sin\theta$

This is what we got from the continuity of B_z