a) Magnetostatics

Maxwell equations

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\partial_t \vec{B} \qquad \vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho \qquad \vec{\nabla} \times \vec{B} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\partial_t \vec{E}$$

For statics we ignore the time derivatives. Then the eqs for $\vec{E}$ and $\vec{B}$ decouple:

Magnetostatics
$$\vec{\nabla} \cdot \vec{B} = 0$$
$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c}\vec{j}$$

current density

b) Current Density

$$\vec{j} = \rho\vec{v}$$

$\rho\vec{v} \cdot d\vec{S}$

charge that flows through $d\vec{s}$ per second.

$$\int_S d\vec{s} \cdot \vec{j} = -\frac{d}{dt}\int d^3r\, \rho$$

$$\| $$

$$\int d^3r\, \vec{\nabla} \cdot \vec{j}$$

$$\Rightarrow \vec{\nabla} \cdot \vec{j} + \frac{\partial\rho}{\partial t} = 0 \qquad \text{continuity equation}$$

statics $\partial_t \rho = 0 \Rightarrow \vec{\nabla} \cdot \vec{j} = 0$

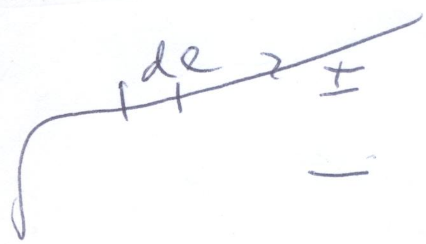
<u>Ohm's law</u> $j_i = \underbrace{\sigma_{ij}}_{\text{conductivity tensor}} E_j$

<u>Magnetic fields</u>;

1820 Oersted discovers that a current creates a magnetic field.

- no magnetic monopoles
- smallest unit is a dipole

c) <u>Experimental Results</u>



$$d\vec{F} = \frac{I}{c}(d\vec{\ell} \times \vec{B})$$

— force of current element

— field creates by a current

$$d\vec{B} = \frac{I}{c} d\vec{\ell} \times \frac{(\vec{x} - \vec{x}_\ell)}{|\vec{x} - \vec{x}_\ell|^3}$$

— Force on charge

$$I = \frac{dq}{dt} \Rightarrow I d\vec{\ell} = dq \frac{d\vec{\ell}}{dt} = dq\, \vec{v}$$

$$\Rightarrow d\vec{F} = \frac{dq}{c} \vec{v} \times \vec{B}$$

$\vec{F} = \frac{q}{c}\,\vec{v} \times \vec{B}$ Lorentz force

Field from a single charge with velocity $\vec{v}$

$$\vec{dB} = \frac{dq}{c}\,\vec{v} \times \frac{(\vec{r} - \vec{r}_q)}{|\vec{r} - \vec{r}_q|^3}$$

$$(dB)_i = \frac{dq}{c}\,\varepsilon_{ijk}\, v_j\, \frac{(\vec{r} - r_q)_k}{|\vec{r} - r_q|^3}$$

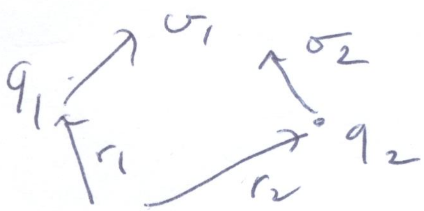
$$\partial_k \frac{1}{|\vec{r} - \vec{r}_q|}$$

$$\Rightarrow \partial_i\, dB_i = \frac{dq}{c}\,\varepsilon_{ijk}\, v_j\, \underbrace{\partial_i \partial_k}_{\text{symmetric}} \frac{1}{|\vec{r} - \vec{r}_q|}$$

anti-symmetric

$$\Rightarrow \vec{\nabla}\,\vec{B} = 0.$$

d) Newton's 3rd law



$$\vec{F}_{21} = \frac{q_1}{c}\,\vec{v}_1 \times \vec{B}_2(r_1)$$

$$= \frac{q_1}{c}\,\vec{v}_1 \times \frac{q_2}{c}\frac{(\vec{v}_2 \times (\vec{r}_1 - \vec{r}_2))}{|\vec{r}_1 - \vec{r}_2|^3}$$

$$\vec{F}_{12} = \frac{q_2}{c}\, v_2\, B_1(r_2)$$

$$= \frac{q_2}{c}\,\vec{v}_2 \times \frac{q_1}{c}\frac{(v_1 \times (\vec{r}_2 - \vec{r}_1))}{(r_1 - \vec{r}_2)^3}$$

Newton's 3rd law is valid if $\vec{F}_{12} = -\vec{F}_{21}$ (34)

This is the case if

$$-\vec{v}_1 \times (\vec{v}_2 \times (\vec{r}_1 - \vec{r}_2)) = \vec{v}_2 \times (\vec{v}_1 \times (\vec{r}_2 - r_1))$$

$$\Rightarrow \quad \vec{v}_1 \times (v_2 \times \underbrace{(\vec{r}_1 - \vec{r}_2)}_{r})) = (\vec{v}_2 \times (\vec{v}_1 \times \underbrace{(\vec{r}_1 - \vec{r}_2)}_{\vec{r}}))$$

outer product satisfies Jacobi identity

$$\vec{v}_1 \times (\vec{v}_2 \times \vec{r}) + \vec{v}_2 \times (\vec{r} \times \vec{v}_1) + \vec{r} \times (\vec{v}_1 \times \vec{v}_2) \Rightarrow$$

(it is not associative

$$- v_2 \times (\vec{v}_1 \times \vec{r})$$

$$\Rightarrow \quad \vec{r} \times (\vec{v}_1 \times \vec{v}_2) = 0.$$

Generally this is not true and Newton III is not valid. The reason is that the magnetic field contains momentum.

e) $\vec{B}$ is a pseudovector

$$\vec{B} = \frac{q}{c} \vec{v} \times \frac{\vec{r}}{r^2}$$

$\vec{v} \to -\vec{v}$ under space inversion.

$r \to -\vec{r}$ under space inversion

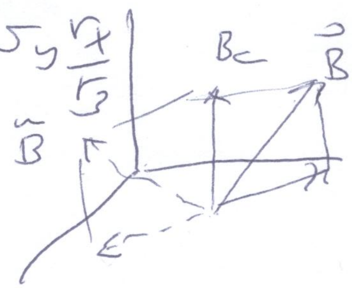
$\Rightarrow \vec{B} \to \vec{B}$ under space inversion

$\vec{B}$ is a pseudovector

Behavior of $\vec{B}$ under reflections

Say reflection in xy plan
then $z \to -z$ $\quad v_z \to -v_z$

$$B_z = \frac{q}{c} v_x \frac{r_y}{r^3} - \frac{q}{c} v_y \frac{r_x}{r^3}$$



$\Rightarrow B_z \to B_z$

$B_x \to -B_x$

$B_y \to -B_y$

Ampère's law

Next we derive Ampères law

$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{J}$$

$\vec{J} \cdot d\vec{A} = I$

$\vec{J} \cdot d\vec{A}\, dl = I\, dl$

choos $d\vec{A} \parallel \vec{J}$

$\vec{J}\, d^3r = I\, \vec{dl}$

$$\Rightarrow d\vec{B} = \vec{J} \frac{d^3r'}{c} \times \vec{\nabla}_r \frac{1}{|\vec{r} - \vec{r}'|}$$

$$\Rightarrow \vec{B} = \int d^3r' \frac{\vec{J}}{c} \times \vec{\nabla}_r \frac{1}{|\vec{r} - \vec{r}'|}$$

$$\vec{\nabla} \times \vec{B} = \int d^3r' \left(\frac{\vec{\nabla} \cdot \vec{J}}{c}\, \vec{\nabla} \frac{1}{|\vec{r} - \vec{r}'|} - \frac{\vec{J}}{c}\, \nabla_r \frac{1}{|\vec{r} - \vec{r}'|} \right.$$

statics $\qquad \vec{\nabla}\cdot\vec{j} + \underbrace{\frac{\partial\rho}{\partial t}}_{=0} = 0 \Rightarrow \vec{\nabla}\cdot\vec{j} = 0$

$$\vec{\nabla}_r \frac{1}{|\vec{r}-r'|} = -4\pi\,\delta^3(r-r')$$

$$\Rightarrow \vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\,\vec{j}$$

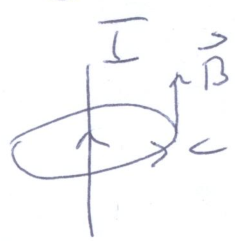
integral form of Ampère's law:

$$\int_S \vec{\nabla}\times\vec{B}\cdot d\vec{s} = \frac{4\pi}{c}\int_S \vec{j}\cdot d\vec{s}$$

$$\parallel$$

$$\oint_C \vec{B}\cdot d\vec{l}$$

$$\Rightarrow \oint_C \vec{B}\cdot dl = \frac{4\pi}{c}\sum I_{\text{enclosed}}.$$



$$B\,2\pi r = \frac{4\pi}{c} I$$

Gauge Potential

$$\vec{B}_i = \frac{1}{c}\int d^3r'\, \vec{j}(\vec{r}') \times (-\vec{\nabla}_r) \frac{1}{|\vec{r}-\vec{r}'|}$$

$$= \frac{1}{c}\int d^3r'\, \varepsilon_{ijk}\, j_j\, \partial_k \frac{-1}{|r-r'|}$$

$$= \frac{1}{c}\,\varepsilon_{ijk}\,\partial_k \int d^3r'\, j_j \frac{-1}{|r-r'|}$$

$$= +\frac{1}{c}\,\varepsilon_{ikj}\,\partial_k \int d^3r'\, j_j \frac{1}{|\vec{r}-r'|}$$

$$= +\,\vec{\nabla} \times \underbrace{\frac{1}{c}\int d^3r'\, \vec{j}\, \frac{1}{|r-r'|}}_{\vec{A}\ \text{gauge potential}}$$

$$\vec{A} = \frac{1}{c}\int d^3r'\, \vec{j}(r') \frac{1}{|r-r'|}$$

$\vec{A}$ is not unique $\quad \vec{A} \to \vec{A} + \vec{\nabla}\varphi$
gives the same $\vec{B}$ field because $\vec{\nabla} \times \vec{\nabla}\varphi = 0$

This is called gauge invariance

gauge theories: em force
strong force
weak force

Example: vector potential of constant $\vec{B}$ field $\quad \vec{B} = (0, 0, B_z)$

$A_1 = B_0\,(0, x, 0)$

$A_2 = B_0(-y, 0, 0)$

$A_3 = \frac{B_0}{2}(-y\, x, 0)$

$$A_1 - A_3 = B_0\left(\frac{y}{2}, \frac{x}{2}, 0\right) = \frac{B_0}{2}\vec{\nabla}(xy)$$

Multipole expansion

Since $A(r) = \frac{1}{c}\int d^3r' \frac{\vec{j}(r')}{|r-r'|}$

we can again do the multipole expansion

$$\frac{1}{|\vec r - r'|} = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \frac{r'^{\ell}}{r^{\ell+1}}\frac{4\pi}{2\ell+1} Y_{\ell m}(\theta,\varphi)\, Y^*_{\ell m}(\theta',\varphi')$$

$$\Rightarrow \bar A(r) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \frac{4\pi}{r^{\ell+1}(2\ell+1)} Y_{\ell m}(\theta,\varphi)\int d^3r'\, r'^{\ell}\, Y^*_{\ell m}(\theta',\varphi') \times \vec j(r')$$

regular product

$\ell=0$ then $\int d^3r'\, \vec j(r')$

consider $j_1 = -\int d^3x\, x_1 \partial_1 j_1$

$\vec\nabla \vec j = 0$ $\quad = -\int d^3x\, x_1(-\partial_2 j_2 - \partial_3 j_3)$

total derivatives

$= 0$

$\Rightarrow$ vector potential of current $\propto \frac{1}{r^2}$

$$\sum_{m=-\ell}^{\ell} Y_{\ell m}(\theta,\varphi)\, Y^*_{\ell m}(\theta',\varphi') \overset{\ell=1}{=}$$

$$\frac{3}{8\pi}\frac{(x+iy)}{r}\frac{(x'-iy')}{r'} + \frac{3}{4\pi}\frac{z z'}{rr'} + \frac{3}{8\pi}\frac{(x-iy)}{r}\frac{(x'+iy')}{r'}$$

$$= -\frac{3}{8\pi}\frac{1}{rr'}\left(xx'+yy' - \cancel{ixy'} + \cancel{iyx'} + 2zz' + xx' + yy' + \cancel{ixy'} - \cancel{iyx'}\right)$$

$$= -\frac{3}{4\pi}\frac{1}{rr'}\left(xx'+yy'+zz'\right) = +\frac{3}{4\pi}\frac{\vec r\cdot\vec r'}{|\vec r|\,|\vec r'|}$$

(44)

So the gauge potential of the $\ell=1$ multipol is given by

$$\vec{A}^{\ell=1}(r) = \frac{4\pi}{3}\frac{1}{r^2}\left(+\frac{3}{4\pi}\right)\int d^3r'\, r'\frac{(\vec{r}\cdot\vec{r}')}{r\,r'}\,\vec{j}(r')$$

$$= +\frac{1}{r^3}\int d^3r'(\vec{r}\cdot\vec{r}')\,\vec{j}(r')$$

$$-\frac{\vec{r}}{r^3} = \vec{\nabla}_r\frac{1}{r} \qquad = -\int d^3r'\,\vec{\nabla}\frac{1}{r}\cdot\vec{r}'\,\vec{j}(r')$$

$$\nabla\vec{j} = 0$$

we want to rewrite this in term of a magnetic moment density $\vec{M} = \frac{\vec{r}\times\vec{j}}{2c}$

magnetic moment $\vec{m} = \int d^3r'\,\frac{\vec{r}'\times j(r')}{2c}$

Let us calculate $\vec{\nabla}\frac{1}{r}\times\vec{m} =$

$$= \frac{1}{2c}\vec{\nabla}\frac{1}{r}\times\int d^3r'\,(r'\times j)$$

$$= \frac{1}{2c}\int d^3r'\,\underbrace{\vec{\nabla}\frac{1}{r}\times(r'\times j)}_{-(\vec{\nabla}\frac{1}{r}\cdot r')\vec{j}\,+\,(\vec{\nabla}\frac{1}{r}\cdot j)\vec{r}'}$$

$$\int j_j\,\partial'_j(r'_i r'_k) = -\int \underbrace{\partial'_j j_j}_{=0}\, r'_i r'_k = 0$$

$$\int j_k r'_i + \int j_i r'_k = 0$$

$$\Rightarrow -\frac{1}{2c}\int d^3r'\,\frac{+r_k}{r^3}\,j_k r'_i = \frac{+1}{2c}\int d^3r\,\frac{r_k}{r^3}\,j_i r'_k$$

$\Rightarrow$ 2nd term gives the same result.

$$\Rightarrow \quad \vec{A}^{\ell=1} = \vec{\nabla}\frac{1}{r}\times\vec{m}$$

The $l=1$ magnetic field is given by

$$B_i^{l=1} = \vec{\nabla} \times A^{l=1} = \vec{\nabla} \times \left(\vec{\nabla}\frac{1}{r} \times \vec{m}\right)$$

$$= -\underbrace{\left(\vec{\nabla}^2 \frac{1}{r}\right)}_{=0} \vec{m} + \vec{\nabla}_k\left(\left(\vec{\nabla}_i \frac{1}{r}\right) m_k\right)$$

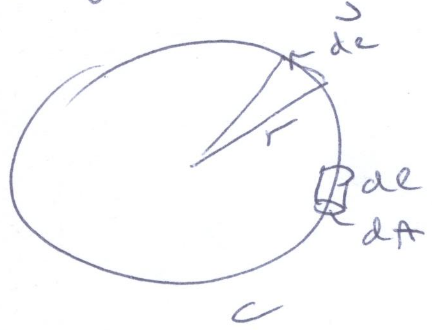
m_k is an integral and does not depend on $\vec{r}$

$$\Rightarrow B_i = +\partial_i\left(\partial_k \frac{1}{r} m_k\right) = -\vec{\nabla}_i \frac{\vec{m}\cdot\vec{r}}{r^3}$$

magnetic potential

Examples

a) magnetic moment of current loop



$$\vec{m} = \frac{1}{2c}\int d^3r\, \vec{r} \times \vec{j}$$

$$\vec{j}\, d^3r = \vec{j}\cdot d\vec{A}\, \vec{dl}$$

$$\vec{m} = \frac{I}{2c}\oint \vec{r} \times \vec{dl} = \frac{IA}{2c}$$ ← area

b) magnetic moment of orbiting point charge

$$\vec{j} = \delta^3(\vec{r}-\vec{r}_p)\, q\, \vec{v}_p$$

$$\vec{m} = \frac{1}{2c}\int d^3r\, \vec{r} \times \delta(\vec{r}-\vec{r}_p)\, q\, \vec{v}_p = \frac{q}{2c}\vec{r}_p \times \vec{v}_p$$

$$= \frac{q}{mc} L_p$$ ← angular momentum

Magnetic fields in matter

<u>model</u> the effect of an external magnetic field can be desribed by a magnetic dipole density $\vec{M}$

Field of a single dipole $\vec{A}_i = -m_i \times \nabla_r \frac{1}{|\vec{r}-\vec{r}_i|}$

$$\Rightarrow A_{tot} = \sum_i -m_i \times \vec{\nabla}_r \frac{1}{|\vec{r}-\vec{r}_i|}$$

$$= -\int d^3r' \, \vec{M}(r') \times \underbrace{\vec{\nabla}_r}_{= -\vec{\nabla}_{r'}} \frac{1}{|r-r'|}$$

$$= -\int d^3r' \left(\vec{\nabla}_{r'} \times \left(\vec{M}(r') \frac{1}{|\vec{r}-\vec{r}'|} \right) - \frac{1}{|r-r'|} \vec{\nabla}_{r'} \times M(r') \right)$$

$$= \int_{\partial V} \frac{1}{|\vec{r}-\vec{r}'|} \vec{M} \times \vec{n} \, da + \int_V d^3r' \frac{1}{|r-r'|} \vec{\nabla}_{r'} \times M(r')$$

induced current density: $c \, \vec{\nabla} \times M$

surface current $c \, \vec{M} \times \vec{n}$

$$\Rightarrow \vec{\nabla} \times \vec{B} = \frac{4\pi}{c} (\vec{J} + J_{ind})$$

$$= \frac{4\pi}{c} \left(\vec{J} + c(\vec{\nabla} \times M) + c \, \vec{M} \times \vec{n} \right)$$

$$\Rightarrow \left(\vec{\nabla}\times B - 4\pi\,\vec{\nabla}\times\vec{M}\right) = \frac{4\pi}{c}\vec{j}$$

For induced volume current.

magnetic field $\vec{H} = \vec{B} - 4\pi\vec{M}$

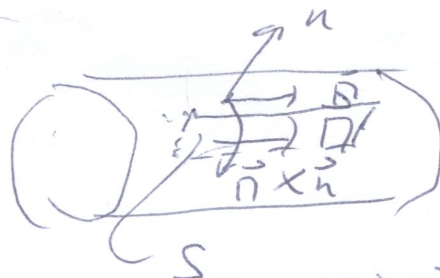
$\vec{j}$ is the current due to the free charges

$\vec{B}$ is called the magnetic induction

For the effect of the surface current we start from the ~~integral~~ integral form of Ampère's law.

$$\int \vec{\nabla}\times\vec{B}\cdot d\vec{s} = \frac{4\pi}{c}\int\vec{j}\cdot d\vec{s} + \frac{4\pi}{c}\int c(\vec{M}\times\vec{n})\,d\vec{s}$$

$$\int \vec{B}\cdot d\vec{l}$$



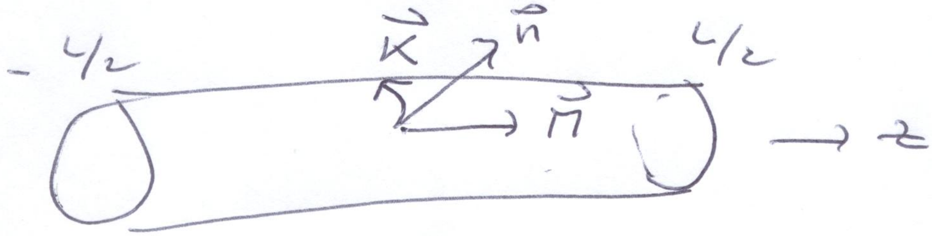
$$\frac{4\pi}{c}\int c(\vec{M}\times\vec{n})\,d\vec{s} = \int 4\pi\,\vec{M}\,d\vec{l}$$

$$\Rightarrow \int(\vec{B} - 4\pi\vec{M})\,dl = \frac{4\pi}{c}\int\vec{j}\cdot d\vec{s}$$

This is important for permanent magnets

Example

permanent magnet in cylinder shape



$$\vec{M}(r) = M\hat{z} = \text{constant}$$

$$\Rightarrow \quad c\,\vec{\nabla}\times\vec{M} = 0 \text{ in bulk.}$$

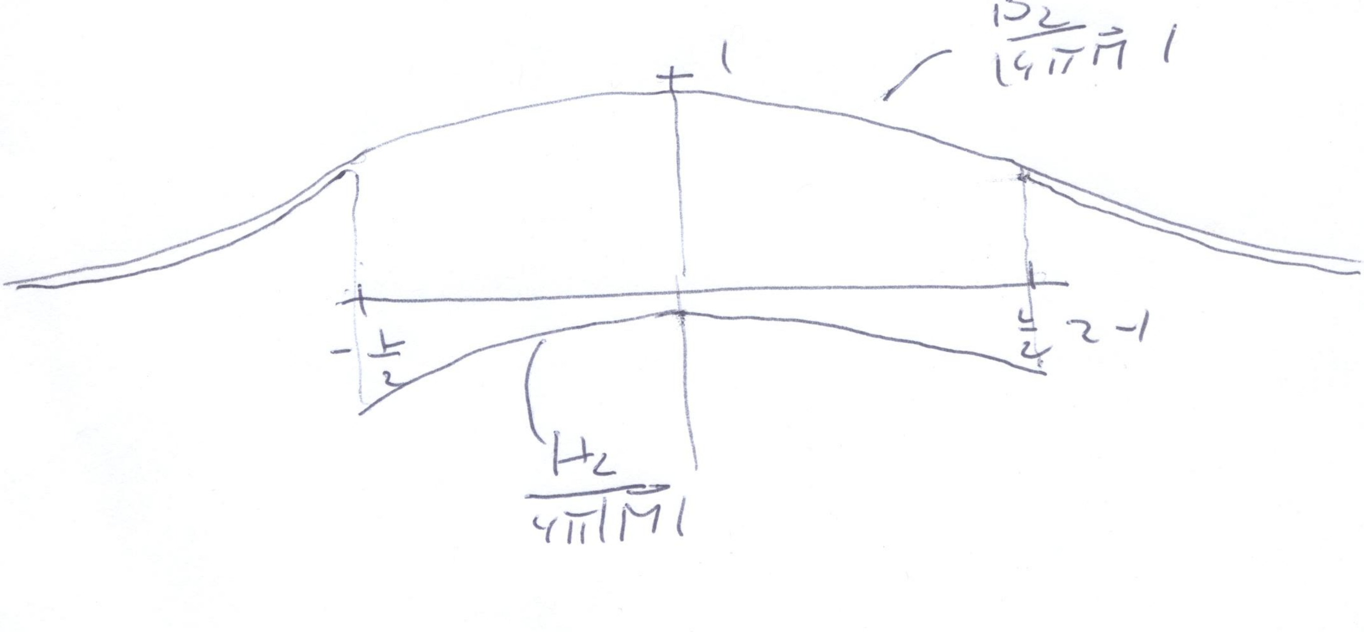
surface current $\quad K = c\,\vec{M}\times\vec{n}$

$$\vec{j} = 0 \Rightarrow \vec{\nabla}\times\vec{H} = 0$$

$$\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\left(\underbrace{c\,\vec{\nabla}\times\vec{M}}_{=0} + c\,\vec{M}\times\vec{n}\right)$$

$$|z| < \frac{L}{2} \qquad \frac{H_z}{4\pi|\vec{M}|} = \frac{|B_z|}{4\pi|\vec{M}|} - 1$$

$$|z| > \frac{L}{2} \qquad \frac{H_z}{4\pi\vec{M}} = \frac{B_z}{4\pi M}$$



Magnets

a) **Permanent Magnets**

material with fixed magnetic dipole density $\vec{M}(\vec{r})$

$$\vec{J}_{free} = 0 \quad \Rightarrow \quad \vec{\nabla} \times \vec{H} = 0$$

always. $\vec{\nabla} \cdot \vec{B} = 0$

$$\Rightarrow \vec{\nabla} \cdot \vec{H} = 4\pi \left(-\vec{\nabla} \cdot \vec{M} \right)$$

$\Rightarrow -\vec{\nabla} \cdot \vec{M}$ is the effective monopole density

$$\vec{\nabla} \times \vec{H} = 0 \quad \Rightarrow \quad \exists \; \phi_M \;|\; \vec{H} = -\vec{\nabla} \phi_M$$

$$\vec{\nabla}^2 \phi_M = 4\pi \; \vec{\nabla} \cdot \vec{M}$$

However the surface current discussed in the previous section is the most important

b) **Paramagnets**

atoms with a permanent magnetic dipole moment $\vec{\mu}$

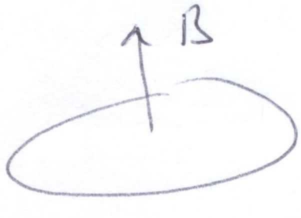
random for $\vec{B}_0 = 0$

oriented for $\vec{B}_0 \neq 0$

$$\vec{B} = \mu \vec{H}$$

$\mu > 1$ because the magnetic moments contribute to $\vec{B}$

c) <u>Diamagnetism</u> $\vec{\mu} = 0$



$\vec{\mu}_d = \dfrac{e\vec{L}}{2mc}$ when we have $\vec{L} = 0$

Lenz : electrons try to oppose an increasing $\vec{B}$

$\Rightarrow$ μ induced is opposite to $\vec{B}$

Generally this is a very small effect