

# a) Magnetostatics

Maxwell equations

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c} \partial_t \vec{B}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \cdot \vec{E} = 4\pi \rho$$

$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{j} + \frac{1}{c} \partial_t \vec{E}$$

For statics we ignore the time derivatives. Then the eqs for  $\vec{E}$  and  $\vec{B}$  decouple:

Magnetostatics

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{j}$$

↑  
current density

# b) Current Density

$$\vec{j} = \rho \vec{v}$$

$$\rho \vec{v} \cdot d\vec{S}$$



charge that flows through  $d\vec{S}$  per second.



$$\int_{S \parallel} d\vec{s} \cdot \vec{j} = -\frac{d}{dt} \int d^3r \rho$$

$$\int d^3r \vec{\nabla} \cdot \vec{j}$$

$$\Rightarrow \vec{\nabla} \cdot \vec{j} + \frac{\partial \rho}{\partial t} = 0$$

continuity equation

statics  $\partial_t \rho = 0 \Rightarrow \vec{\nabla} \cdot \vec{j} = 0$

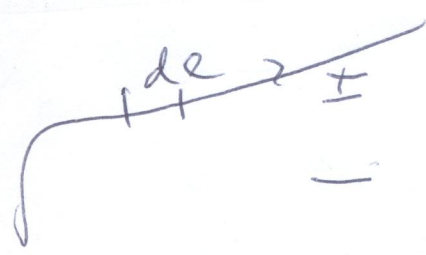
Ohm's Law  $j_i = \sigma_{ij} E_j$   
↑  
conductivity tensor

Magnetic Fields;

1820 Oersted discovers that a current creates a magnetic field.

- no magnetic monopoles
- smallest unit is a dipole

c) Experimental Results



$d\vec{F} = \frac{I}{c} (d\vec{l} \times \vec{B})$   
— force of current element

— field created by a current

$$d\vec{B} = \frac{I}{c} d\vec{l} \times \frac{(\vec{x} - \vec{x}_e)}{|\vec{x} - \vec{x}_e|^3}$$

— Force on charge

$$I = \frac{dq}{dt} \Rightarrow I d\vec{l} = dq \frac{d\vec{l}}{dt} = dq \vec{v}$$

$$\Rightarrow d\vec{F} = \frac{dq}{c} \vec{v} \times \vec{B}$$

$$\vec{F} = \frac{q}{c} \vec{v} \times \vec{B}$$

Lorentz force

(38)

Field from a single charge with velocity  $\vec{v}$

$$d\vec{B} = \frac{dq}{c} \vec{v} \times \frac{(\vec{r} - \vec{r}_q)}{|\vec{r} - \vec{r}_q|^3}$$

$$(dB)_i = \frac{dq}{c} \epsilon_{ijk} v_j \frac{(\vec{r} - \vec{r}_q)_k}{|\vec{r} - \vec{r}_q|^3}$$

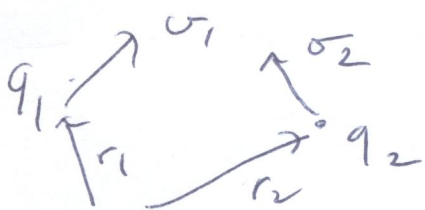
$$\partial_k \frac{1}{|\vec{r} - \vec{r}_q|}$$

$$\Rightarrow \partial_i dB_i = \frac{dq}{c} \epsilon_{ijk} v_j \underbrace{\partial_i \partial_k}_{\text{symmetric}} \frac{1}{|\vec{r} - \vec{r}_q|}$$

↑  
anti-symmetric

$$\Rightarrow \vec{\nabla} \cdot \vec{B} = 0.$$

d) Newton's 3rd law



$$\vec{F}_{21} = \frac{q_1}{c} \vec{v}_1 \times \vec{B}_2(r_1)$$

$$= \frac{q_1}{c} \vec{v}_1 \times \frac{q_2 (\vec{v}_2 \times (\vec{r}_1 - \vec{r}_2))}{|\vec{r}_1 - \vec{r}_2|^3}$$

$$\vec{F}_{12} = \frac{q_2}{c} \vec{v}_2 \times \vec{B}_1(r_2)$$

$$= \frac{q_2}{c} \vec{v}_2 \times \frac{q_1 (\vec{v}_1 \times (\vec{r}_2 - \vec{r}_1))}{|\vec{r}_1 - \vec{r}_2|^3}$$



Newton's 3rd law is valid if  $\vec{F}_{12} = -\vec{F}_{21}$  (By)

This is the case if

$$-\vec{v}_1 \times (\vec{v}_2 \times (\vec{r}_1 - \vec{r}_2)) = \vec{v}_2 \times (\vec{v}_1 \times (\vec{r}_2 - \vec{r}_1))$$

$$\Rightarrow \vec{v}_1 \times (\vec{v}_2 \times (\vec{r}_1 - \vec{r}_2)) = (\vec{v}_2 \times (\vec{v}_1 \times (\vec{r}_2 - \vec{r}_1)))$$

outer product satisfies Jacobi identity

$$\vec{v}_1 \times (\vec{v}_2 \times \vec{r}) + \vec{v}_2 \times (\vec{r} \times \vec{v}_1) + \vec{r} \times (\vec{v}_1 \times \vec{v}_2) = 0$$

it is not associative

$$-\vec{v}_2 \times (\vec{v}_1 \times \vec{r})$$

$$\Rightarrow \vec{r} \times (\vec{v}_1 \times \vec{v}_2) = 0$$

Generally this is not true and Newton III is not valid. The reason is that the magnetic field contains momentum.

e)  $\vec{B}$  is a pseudovector

$$\vec{B} = \frac{q}{c} \vec{v} \times \frac{\vec{r}}{r^2}$$

$\vec{v} \rightarrow -\vec{v}$  under space inversion.

$$\vec{r} \rightarrow -\vec{r}$$

$\Rightarrow \vec{B} \rightarrow \vec{B}$  under space inversion

$\vec{B}$  is a pseudovector

# Behavior of $\vec{B}$ under reflections 340

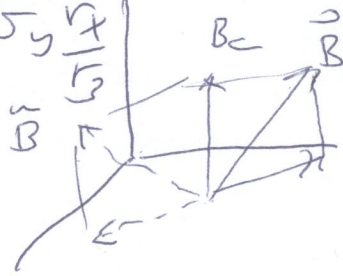
Say reflection in  $xy$  plane  
 then  $z \rightarrow -z$        $v_z \rightarrow -v_z$

$$B_z = \frac{q}{c} v_x \frac{y}{r_0} - \frac{q}{c} v_y \frac{x}{r_0}$$

$$\Rightarrow B_z \rightarrow B_z$$

$$B_x \rightarrow -B_x$$

$$B_y \rightarrow -B_y$$



## Ampère's Law

Next we derive Ampère's Law

$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{j}$$



$$\vec{j} \cdot d\vec{A} = I$$

$$\vec{j} \cdot d\vec{A} dl = I dl$$

cross  $d\vec{A} \parallel \vec{j}$

$$\vec{j} \cdot d^3r = I dl$$

$$\Rightarrow d\vec{B} = \frac{1}{c} \int d^3r' \times \vec{\nabla}_r \frac{1}{|\vec{r} - \vec{r}'|}$$

$$\Rightarrow \vec{B} = \int d^3r' \vec{j} \times \vec{\nabla}_r \frac{1}{|\vec{r} - \vec{r}'|}$$

$$\vec{\nabla} \times \vec{B} = \int d^3r' \left( \frac{\vec{\nabla} \cdot \vec{j}}{c} \frac{1}{|\vec{r} - \vec{r}'|} - \frac{\vec{\nabla} \cdot \vec{j}}{c} \vec{\nabla}_r \frac{1}{|\vec{r} - \vec{r}'|} \right)$$

statics

$$\vec{\nabla} \cdot \vec{j} + \frac{\partial \rho}{\partial t} = 0 \Rightarrow \vec{\nabla} \cdot \vec{j} = 0$$

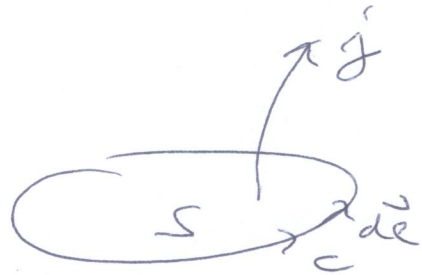
$$\vec{\nabla} \cdot \frac{1}{|\vec{r} - \vec{r}'|} = -4\pi \delta^3(\vec{r} - \vec{r}')$$

$$\Rightarrow \vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{j}$$

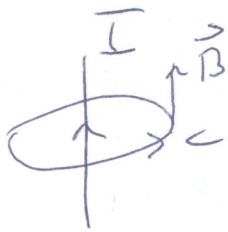
integral form of Ampère's Law.

$$\oint_S \vec{\nabla} \times \vec{B} \cdot d\vec{s} = \frac{4\pi}{c} \int_S \vec{j} \cdot d\vec{s}$$

$$\oint_C \vec{B} \cdot d\vec{e}$$



$$\Rightarrow \oint_C \vec{B} \cdot d\vec{e} = \frac{4\pi}{c} \sum \vec{I} \text{ enclosed.}$$



$$B 2\pi r = \frac{4\pi}{c} I$$



# Gauge Potential

$$\vec{B}_i = \frac{1}{c} \int d^3 r' \vec{j}(r') \times (\vec{\nabla}_r) \frac{1}{|\vec{r} - \vec{r}'|}$$

$$= \frac{1}{c} \int d^3 r' \epsilon_{ijk} j_j \partial_k \frac{1}{|\vec{r} - \vec{r}'|}$$

$$= \frac{1}{c} \epsilon_{ijk} \partial_k \int d^3 r' j_j \frac{1}{|\vec{r} - \vec{r}'|}$$

$$= + \frac{1}{c} \epsilon_{cij} \partial_k \int d^3 r' j_j \frac{1}{|\vec{r} - \vec{r}'|}$$

$$= + \vec{\nabla} \times \frac{1}{c} \int d^3 r' \vec{j} \frac{1}{|\vec{r} - \vec{r}'|}$$

A gauge potential

$$\vec{A} = \frac{1}{c} \int d^3 r' \vec{j}(r') \frac{1}{|\vec{r} - \vec{r}'|}$$

$\vec{A}$  is not unique  $\vec{A} \rightarrow \vec{A} + \vec{\nabla} \psi$   
gives the same  $\vec{B}$  field because  $\vec{\nabla} \times \vec{\nabla} \psi = 0$

This is called gauge invariance

gauge theories:

- em force
- strong force
- weak force

Example: vector potential of constant  $\vec{B}$  field  $\vec{B} = (0, 0, B_z)$

$$\begin{aligned} A_1 &= B_z (0, x, 0) \\ A_2 &= B_z (-y, 0, 0) \\ A_3 &= \frac{B_z}{2} (-y x, 0) \end{aligned}$$

$$\begin{aligned} A_1 - A_3 &= B_z \left( \frac{y}{2}, \frac{x}{2}, 0 \right) \\ &= \frac{B_z}{2} \vec{\nabla} (xy) \end{aligned}$$

# Multipole expansion

Since  $A(r) = \frac{1}{c} \int d^3r' \frac{\vec{j}(r')}{|r-r'|}$



we can again do the multipole expansion

$$\frac{1}{|r-r'|} = \sum_{l=0}^{\infty} \sum_{m=-l}^l \frac{r'^l}{r^{l+1}} \frac{4\pi}{2l+1} Y_{lm}(\theta, \varphi) Y_{lm}^*(\theta', \varphi')$$

\$\Rightarrow \vec{A}(r) = \sum\_{l=0}^{\infty} \sum\_{m=-l}^l \frac{4\pi}{r^{l+1}(2l+1)} Y\_{lm}(\theta, \varphi) \int d^3r' r'^l Y\_{lm}^\*(\theta', \varphi') \times \vec{j}(r')\$

regular product

\$l=0\$ then  $\int d^3r' \vec{j}(r')$

consider  $\vec{j}_1 = -\int d^3x x_1 \partial_1 \dot{j}_1$

$\vec{\nabla} \cdot \vec{j} = 0 = -\int d^3x x_1 (-\partial_2 \dot{j}_2 - \partial_3 \dot{j}_3)$

total derivative

= 0

\$\Rightarrow\$ vector potential of current is  $\frac{1}{r^2}$

$$\sum_{m=-l}^l Y_{lm}(\theta, \varphi) Y_{lm}^*(\theta', \varphi') =$$

$$\frac{3}{8\pi} \frac{(x+iy)(x'-iy')}{r^2} + \frac{3}{4\pi} \frac{2z}{r^2} - \frac{3}{8\pi} \frac{(x-iy)(x'+iy')}{r^2}$$

$$= -\frac{3}{8\pi} \frac{1}{r^2} (xx' + yy' - ixy' + iyx') + \frac{3}{4\pi} \frac{2zz'}{r^2} + \frac{3}{8\pi} \frac{1}{r^2} (xx' + yy' + ixy' - iyx')$$

$$= -\frac{3}{4\pi} \frac{1}{r^2} (xx' + yy' + 2z') = -\frac{3}{4\pi} \frac{\vec{r} \cdot \vec{r}'}{r^2 |\vec{r}'|^2}$$



So the gauge potential of the  $l=1$  multipole <sup>(44)</sup> is given by

$$\vec{A}^{l=1}(\vec{r}) = \frac{4\pi}{3} \frac{1}{r^2} \left(-\frac{3}{4\pi}\right) \int d^3r' r' \frac{(\vec{r} \cdot \vec{r}')}{r r'} \vec{j}(\vec{r}')$$

$$= -\frac{1}{r^3} \int d^3r' (\vec{r} \cdot \vec{r}') \vec{j}(\vec{r}')$$

$$-\frac{\vec{r}}{r^3} = \vec{\nabla}_r \frac{1}{r} \quad = \int d^3r' \vec{\nabla} \frac{1}{r} \cdot \vec{r}' \vec{j}(\vec{r}')$$

$$\vec{\nabla} \cdot \vec{j} = 0$$

we want to rewrite this in terms of a magnetic moment density  $\vec{M} = \frac{\vec{r} \times \vec{j}}{2c}$

$$\text{magnetic moment } \vec{m} = \int d^3r' \frac{\vec{r}' \times \vec{j}(\vec{r}')}{2c}$$

$$\text{Let us calculate } \vec{\nabla} \frac{1}{r} \times \vec{m} =$$

$$= \frac{1}{2c} \vec{\nabla} \frac{1}{r} \times \int d^3r' (\vec{r}' \times \vec{j})$$

$$= \frac{1}{2c} \int d^3r' \underbrace{\vec{\nabla} \frac{1}{r} \times (\vec{r}' \times \vec{j})}_{(\vec{\nabla} \frac{1}{r} \cdot \vec{r}') \vec{j}' - (\vec{\nabla} \frac{1}{r} \cdot \vec{j}) \vec{r}'}$$

$$\int \delta_j \delta_j (\vec{r}' \cdot \vec{r}') = - \int \delta_j \delta_j r_i r_i = 0$$

$$\int \delta_k r_i' + \int \delta_i r_k' = 0$$

$$\Rightarrow -\frac{1}{2c} \int d^3r' + \frac{r_k}{r^3} \delta_k r_i' = +\frac{1}{2c} \int d^3r' \frac{r_k}{r^3} \delta_i r_k'$$

$\Rightarrow$  2nd term gives the same result.

$$\Rightarrow \vec{A}^{l=1} = \vec{\nabla} \frac{1}{r} \times \vec{m}$$

The  $l=1$  magnetic field is given by

(45)

$$\begin{aligned} \vec{B}^{l=1} &= \vec{\nabla} \times \vec{A}^{l=1} = \vec{\nabla} \times \left( \vec{\nabla} \frac{1}{r} \times \vec{m} \right) \\ &= \underbrace{\left( \vec{\nabla} \frac{1}{r} \right)}_{\substack{\parallel \\ 0}} \vec{m} + \vec{\nabla}_\kappa \left( \left( \vec{\nabla} \frac{1}{r} \right)_\kappa m_\kappa \right) \end{aligned}$$

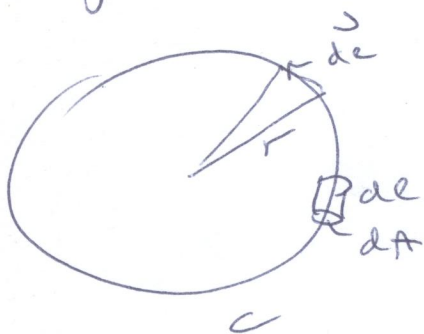
$m_\kappa$  is an integral and does not depend on  $\vec{r}$

$$\Rightarrow B_i = +\partial_i \left( \partial_\kappa \frac{1}{r} m_\kappa \right) = -\partial_i \frac{\vec{m} \cdot \vec{r}}{r^3}$$

↑  
magnetic potential

### Examples

a) magnetic moment of current loop



$$\vec{m} = \frac{1}{2c} \int d^3r \vec{r} \times \vec{j}$$

$$\vec{j} d^3r = \vec{j} \cdot d\vec{A} d\ell$$

$$\vec{m} = \frac{I}{2c} \oint \vec{r} \times d\vec{\ell} = \frac{IA}{2c} \leftarrow \text{area}$$

b) magnetic moment of orbiting point charge

$$\vec{j} = \delta(\vec{r} - \vec{r}_p) q \vec{v}_p$$

$$\begin{aligned} \vec{m} &= \frac{1}{2c} \int d^3r \vec{r} \times \delta(\vec{r} - \vec{r}_p) q \vec{v}_p = \frac{q}{2c} \vec{r}_p \times \vec{v}_p \\ &= \frac{q}{mc} L_p \leftarrow \text{angular momentum} \end{aligned}$$