

Electric dipole radiation

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harmonic source

$$\vec{A} = \frac{e^{-i\omega(t-\frac{r}{c})}}{cr} \int d^3r' e^{-\frac{i\omega}{c} \hat{r} \cdot \vec{r}'} \vec{j}_0(r')$$

$$\phi = \frac{e^{-i\omega(t-\frac{r}{c})}}{r} \int d^3r' e^{-\frac{i\omega}{c} \hat{r} \cdot \vec{r}'} \rho(r')$$

$$\omega = kc = \frac{2\pi c}{\lambda} \Rightarrow \frac{\omega}{c} = \frac{2\pi}{\lambda}$$

wavelength much larger than the size of the source then $|\frac{\omega}{c} \hat{r} \cdot \vec{r}'| \ll 1$

$$e^{-\frac{i\omega}{c} \hat{r} \cdot \vec{r}'} = 1 - \frac{i\omega}{c} \hat{r} \cdot \vec{r}'$$

↑ to leading order we only keep this term

$$\int d^3r' j_1(r') = - \int d^3r' x_1 \partial_1 j_1(r')$$

↑ one component

$$\int d^3r' \partial_2 j_2 = 0$$
$$\int d^3r' \partial_3 j_3 = 0$$

$$= \int d^3r' \partial_t \rho x_1 = \partial_t \int d^3r' \rho x_1$$

$$= \partial_t P_1 \leftarrow \text{one component of dipole moment}$$

$$\Rightarrow \vec{A} = \frac{e^{-i\omega(t-\frac{r}{c})}}{cr} (-i\omega) \vec{P}$$
$$\vec{B} = \vec{\nabla} \times \vec{A} = \frac{i\omega}{c^2} \hat{r} \times \vec{P} e^{-i\omega(t-\frac{r}{c})} \frac{(-i\omega)}{r}$$

$$\vec{B} = \frac{\omega^2}{rc^2} \hat{r} \times \vec{p} e^{-i\omega(t-\frac{r}{c})}$$

away from the sources we have $\vec{j} = 0$

$$\Rightarrow \partial_t E = c \nabla \times \vec{B} - 4\pi \vec{j}$$

↓
0

$$\Rightarrow E = \frac{ic}{\omega} \nabla \times \vec{B}$$

Far away from the source (radiation zone) we have

$$\begin{aligned} \vec{E} &= \frac{ic}{\omega} \frac{i\omega}{c} \hat{r} \times (\hat{r} \times \vec{p}) e^{-i\omega(t-\frac{r}{c})} \\ &= -\frac{\omega^2}{rc^2} (\hat{r} \cdot \vec{p}) \hat{r} - \vec{p} e^{-i\omega(t-\frac{r}{c})} \end{aligned}$$

$$\Rightarrow \vec{B} = \hat{r} \times \vec{E}$$

$$\vec{E} \cdot \hat{r} = 0$$



Poynting vector $\vec{P} = \frac{c}{4\pi} \text{Re}(\vec{E} \times \vec{B}^*)$

$$\begin{aligned} &= \frac{c}{4\pi} \text{Re}(\vec{E} \times (\hat{r} \times \vec{E}^*)) \\ &= \frac{c}{4\pi} \hat{r} (|\vec{E}|^2) \end{aligned}$$

$$\begin{aligned} \frac{dW}{dt d\Omega} &= \vec{P} \cdot \hat{r} r^2 = \frac{c}{4\pi} \frac{r^2 \omega^4}{r^2 c^4} (p^2 - (\hat{r} \cdot \vec{p})^2) \\ &= \frac{c}{4\pi} \frac{\omega^4}{c^4} p^2 \sin^2 \theta \end{aligned}$$

Magnetic dipole radiation

We use the next term in the expansion

$$e^{-i\vec{k}\cdot\vec{r}} = 1 - i\vec{k}\cdot\vec{r}$$

$$\int d^3r' e^{-i\vec{k}\cdot\vec{r}'} j_0(r')$$

$$\int d^3r' (-i\vec{k}\cdot\vec{r}') j_0(r')$$

$$-i k_j r'_j j_0 = -i k_j (x'_j \dot{j}_0 - x'_j \dot{j}_0) / c + (x'_j \dot{j}_0 + x'_j \dot{j}_0) / c$$

This gives

$$\int d^3r (-i\vec{k}\cdot\vec{r}') j_0(r') = \int d^3r' (i\vec{k} \times (\vec{r}' \times \frac{\vec{j}_0(r')}{2})) e$$

$$+ \left(\frac{i k_i}{2} \int d^3r' x'_i x'_e \partial'_i \cdot \vec{j}_0 \right) e$$

Partial integrate
- ∂_t ρ

$$= i\vec{k} \times \vec{P}_M + \frac{\omega k_i}{2} \int d^3r' x'_i x'_e P_0(\vec{r}')$$

magnetic dipole moment

$$\vec{P}_M = \int d^3r'$$

this gives quadrupole radiation

Magnetic dipole radiation

$$\vec{A} = \frac{e^{-i\omega(t-r/c)}}{cr} c \vec{u} \times \vec{P}_M$$

$$\begin{aligned} \vec{B}_M &= \vec{\nabla} \times \vec{A} = i\vec{k} \times A + \mathcal{O}\left(\frac{1}{r^2}\right) \\ &= -\vec{k} \times (\vec{k} \times \vec{P}_M) \frac{e^{-i\omega(t-r/c)}}{r} \end{aligned}$$

$$E_M = \frac{c}{\kappa} \vec{\nabla} \times \vec{B}_M = \frac{c}{\kappa} \vec{u} \times \vec{B}_M + \mathcal{O}\left(\frac{1}{r^2}\right)$$

$$\left(\partial_t \vec{E} = c \vec{\nabla} \times \vec{B} - 4\pi \vec{j}\right) = -\frac{\vec{k}}{\epsilon} \times \vec{B}_M = -\hat{k} \times \vec{B}_M$$

$$\Rightarrow \frac{dw}{dt d\Omega} = \frac{\omega^4}{8\pi c^3} P_M^2 \sin^2 \theta$$

Electric quadrupole radiation

$$\vec{A}_{E2} = \frac{e^{-i\omega(t-r/c)}}{rc} \int d^3r' -\frac{\omega \kappa_i}{2} x_i' x_j' \rho_0(r')$$

$$\vec{B} = \vec{\nabla} \times \vec{A} = i\vec{k} \times \vec{A} + \mathcal{O}\left(\frac{1}{r^2}\right) \quad \text{with } \vec{A} = \frac{2Q_{ij} \vec{r}}{rc}$$

$$\vec{B}_r = \frac{ie}{rc} \epsilon_{pqr} \kappa_q \frac{\omega \kappa_i}{2} \int d^3r' \left(x_i' x_j' - \frac{1}{3} \delta_{ij} r'^2 \right) \rho_0(r', t)$$

$$\vec{B}_{E2} = \frac{e^{-i\omega(t-r/c)}}{rc} \epsilon_{pqr} \kappa_q Q_{ik} \kappa_i$$

vanishes when contracted with ϵ

$$\vec{E}_{E2} = \frac{c}{\kappa} \vec{u} \times \vec{B}_{E2}$$

Q_{ij} is the quadrupole moment

Example radiating point particle

$$r(t) = \text{Re}(\vec{r}_0 e^{-i\omega t}) \Rightarrow \vec{p} = q \vec{r}_0$$

$$\ddot{a}(t) = \text{Re}(\vec{r}_0 (-i\omega)^2 e^{-i\omega t})$$

$$\begin{aligned} \frac{dW}{d\Omega dt} &= \frac{q^2}{4\pi c^3} \overline{\dot{a}^2} \sin^2 \theta \\ &= \frac{q^2}{4\pi c^3} p^2 \omega^4 \sin^2 \theta \overline{\cos^2 \omega t} \\ &= \frac{\omega^4}{8\pi c^3} \dot{p}^2 \sin^2 \theta \end{aligned}$$

which is exactly the result from the dipole approximation.

Special Relativity

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a) General definitions

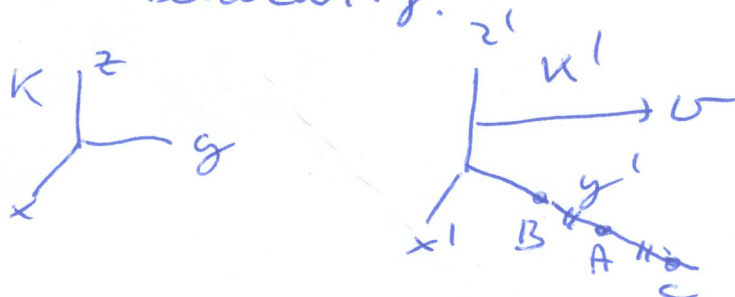
Inertial Frame a frame in which a particle on which no forces are exerted moves with constant velocity

System of reference coordinate system + clock

principle of relativity all laws of physics are identical in all inertial frames

maximum velocity The maximum velocity in all inertial frame is the speed of light. This has been tested with great accuracy by experiments. This is also known as Einsteins principle of relativity

time is absolute in Newtonian dynamics but not in relativity.

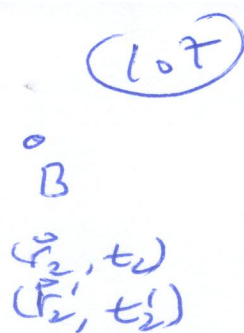
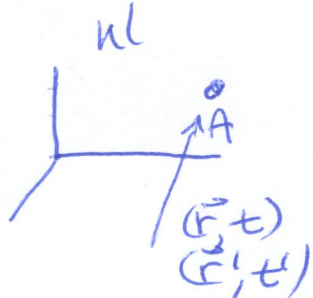


a signal emitted at A arrives simultaneously at B and C in K' but not in K

Event position + time

World point position + time coordinates

World line space time coordinates of a particle as a function of time



If A and B are connected by a light ray then

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 - c^2(t_2 - t_1)^2 = 0$$

$$(x'_1 - x'_2)^2 + (y'_1 - y'_2)^2 + (z'_1 - z'_2)^2 - c^2(t'_2 - t'_1)^2 = 0$$

relativistic interval between two events

$$s_{12} = (c^2(t_2 - t_1)^2 - (\vec{r}_1 - \vec{r}_2)^2)^{1/2}$$

infinitesimally: $ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$

Theorem ds^2 is the same for all inertial frames

ok if $ds = 0$

if $ds \neq 0$ then $(ds)^2 = a(\vec{v}_1, \vec{v}_2) (ds)^2$

$a(\vec{v}_1, \vec{v}_2)$ does not depend on spacetime because of the homogeneity of spacetime. That is why a only depends on \vec{v}_1 and \vec{v}_2 .

consider 3 coordinate systems

K $\vec{v} = 0$ $(ds_1)^2 = a(\vec{v}_1) ds^2$

K_1 \vec{v}_1 $(ds_2)^2 = a(\vec{v}_2) ds^2$

K_2 \vec{v}_2 $(ds_2)^2 = a(\vec{v}_1, \vec{v}_2) (ds_1)^2$

$$\Rightarrow a(\vec{v}_1, \vec{v}_2) = \frac{a(\vec{v}_2)}{a(\vec{v}_1)}$$

homogeneity of space time.

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$\Rightarrow a(v_1, v_2)$ only depends on $(v_1 - v_2)^2$

\Rightarrow Let $a(v_1, v_2) = b((v_1 - v_2)^2)$

$$\Rightarrow b(v_1^2 + v_2^2 - 2v_1v_2 \cos \theta) = \frac{b(v_2^2)}{b(v_1^2)}$$

does not depend on θ

only possible if $b(v^2) = \text{constant}$

$$\Rightarrow a(v_1, v_2) = \text{constant} = \alpha$$

$$a(v_1, v_2) = \frac{a(v_1)}{a(v_2)} \Rightarrow \alpha = \frac{\alpha}{\alpha} = 1$$

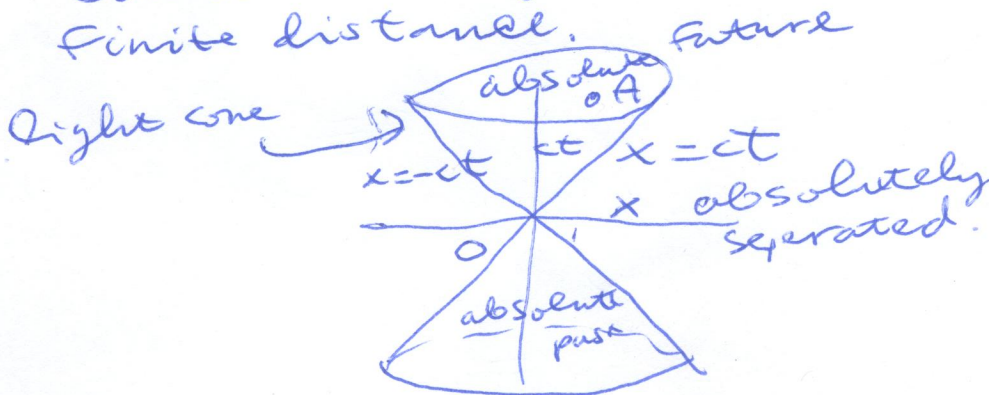
$\Rightarrow (ds)^2$ is the same for all frames

time like intervals intervals for which there exists a frame in which two events occur at the same ^{space} point. Then

$$s_{12}^2 = c^2(t_2 - t_1)^2 - (x_2 - x_1)^2 > 0$$

space like interval $s_{12}^2 = c^2(t_1 - t_2)^2 - (x_1 - x_2)^2 < 0$

\exists frame in which two events occur simultaneously but are separated by a finite distance.



There is no frame for which A and O occur simultaneously \Rightarrow A is in the future for all frames

proper time time by a clock moving with the object

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in $c dt$ the clock moves by (dx, dy, dz)

in K' , frame of the clock, $dx' = dy' = dz' = 0$

$(ds)^2$ is invariant.

$$\Rightarrow c^2 dt^2 - dx^2 - dy^2 - dz^2 = c^2 dt'^2$$

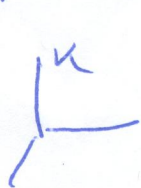
$$\Rightarrow (dt') = dt \left(1 - \frac{dx^2 + dy^2 + dz^2}{c^2 dt^2} \right)^{1/2}$$

$$= dt \sqrt{1 - \frac{v^2}{c^2}}$$

for finite time intervals

$$t'_i - t'_f = \int_{t_i}^{t_f} dt \sqrt{1 - \frac{v^2}{c^2}}$$

proper time is less; moving clocks run slower



- in K clocks in K' lag

- in K' clocks in K lag.

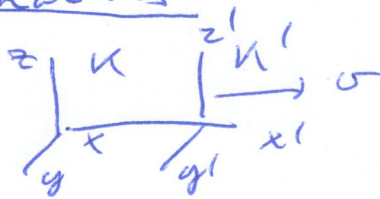
no contradiction because we have to compare clocks in the same inertial frame

Twin paradox



moving twin stays younger
- no contradiction because moving twin is not an inertial frame.

Lorentz transformations



Question: event (x, y, z, t) in K , what are its coordinates in K' ?

in CM $t' = t, y' = y, z' = z, x' = x - vt$

relativity s^2 is the same for all frames

$$x_0 = ct$$

$$x_0' = ct'$$

$$x_1 = x$$

$$x_1' = x'$$

then $x_0^2 - x_1^2 = x_0'^2 - x_1'^2$

Homogeneity of space \Rightarrow relation should be linear

$$\begin{pmatrix} x_0' \\ x_1' \end{pmatrix} = L(\beta) \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \quad \beta = \frac{v}{c}$$

$\begin{pmatrix} a & b \\ d & f \end{pmatrix}$

$$\begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = \begin{pmatrix} x_0' \\ x_1' \end{pmatrix} \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \begin{pmatrix} x_0' \\ x_1' \end{pmatrix}$$

$$= \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} L(\beta)^T \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} L(\beta) \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}$$

$$\forall x_0, x_1 \Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = L(\beta)^T \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} L(\beta)$$

$$= \begin{pmatrix} a^2 - d^2 & ab - df \\ b^2 - f^2 & ab - df \end{pmatrix}$$

$$\Rightarrow ab = fd$$

$$a^2 - d^2 = 1$$

$$b^2 - f^2 = -1$$

for $x_1 = 0$ we have that $x_1' = -\beta x_0'$

$$\begin{pmatrix} x_0' \\ x_0' \beta \end{pmatrix} = \begin{pmatrix} a & b \\ d & f \end{pmatrix} \begin{pmatrix} x_0 \\ 0 \end{pmatrix} = \begin{pmatrix} a x_0 \\ d x_0 \end{pmatrix} \Rightarrow \frac{d}{a} = \beta$$

at $x_1' = 0$ we have that $x_1 = vt = \beta x_0$
 $x_1' = dx_0 + f x_1 \Rightarrow 0 = dx_0 + f \beta x_0$
 $\Rightarrow d + f \beta = 0$

$$\Rightarrow \beta = -\frac{d}{f}$$

$$d + f \frac{d}{f} = 0$$

$$\Rightarrow a = f$$

$$a^2 - d^2 = 1 \Rightarrow a^2 - \beta^2 a^2 = 1$$

$$\Rightarrow a = \frac{1}{\sqrt{1 - \beta^2}}$$

$$\Rightarrow f = \frac{1}{\sqrt{1 - \beta^2}}, \quad d = -\frac{\beta}{\sqrt{1 - \beta^2}}$$

$$b^2 = f^2 - 1 = \frac{1}{1 - \beta^2} - \frac{(1 - \beta^2)}{(1 - \beta^2)} = \frac{\beta^2}{(1 - \beta^2)}$$

$$\Rightarrow b = -\frac{\beta}{\sqrt{1 - \beta^2}}$$

sign follows from nonrelativistic limit

$$\Rightarrow L = \begin{pmatrix} \frac{1}{\sqrt{1 - \beta^2}} & -\frac{\beta}{\sqrt{1 - \beta^2}} \\ -\frac{\beta}{\sqrt{1 - \beta^2}} & \frac{1}{\sqrt{1 - \beta^2}} \end{pmatrix}$$

Useful parameterization $\beta = \tanh \gamma$
↑
rapidity

$$\text{then } \gamma = \frac{1}{\sqrt{1-\beta^2}} = \cosh \gamma$$

$$\beta \gamma = \sinh \gamma$$

$$L = \begin{pmatrix} \cosh \gamma & -\sinh \gamma \\ \sinh \gamma & \cosh \gamma \end{pmatrix}$$

Lorentz transformation

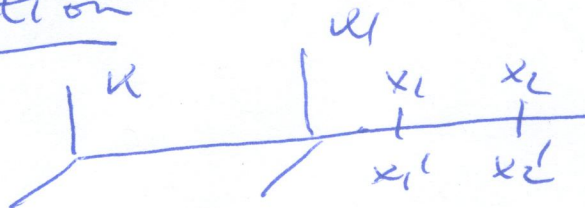
$$x_0' = \gamma x_0 - \beta \gamma x_1$$

$$x_1' = -\beta \gamma x_0 + \gamma x_1$$

$$x_2' = x_2$$

$$x_3' = x_3$$

Lorentz contraction



length of rod in K : $l_0 = x_2 - x_1$
measure the length of the rod at
 $ct_1' = ct_2' = 0$ (the times have to
be the same for the measurement of the rod)

$$v_y = \frac{v_y'}{\gamma \left(1 + \frac{v v_x'}{c^2}\right)}$$

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same for $v_z = \frac{v_z'}{\gamma \left(1 + \frac{v v_x'}{c^2}\right)}$

classical limit

$$v_x = v_x' + v$$

$$v_y = v_y'$$

$$v_z = v_z'$$

$$ct_1' = \gamma(ct_1 - \beta x_1) \Rightarrow ct_1 = \beta x_1$$

$$ct_2' = \gamma(ct_2 - \beta x_2) \Rightarrow ct_2 = \beta x_2$$

$$x_1 = \gamma(-\beta ct_1 + x_1) = \gamma(-\beta^2 x_1 + x_1)$$

$$x_2' = \gamma(-\beta ct_2 + x_2) = \gamma(-\beta^2 x_2 + x_2)$$

$$\Rightarrow x_1' = \frac{x_1}{\gamma} \quad \Rightarrow (x_2' - x_1') = \frac{(x_2 - x_1)}{\gamma}$$

$$x_2' = \frac{x_2}{\gamma} \quad L = \frac{L_0}{\gamma} \leftarrow \text{proper length}$$

Lorentz contraction

for volume: $V = V_0 \sqrt{1 - \beta^2}$

no contraction $\perp \vec{v}$

Addition of velocities

$$v_x = \frac{dx}{dt} \quad v_x' = \frac{dx'}{dt'}$$

$$dx = \gamma(dx' + v dt') \quad dy' = dy \quad dz' = dz$$

$$dt = \gamma(dt' + \frac{v}{c^2} dx')$$

$$\Rightarrow \frac{dt}{dt'} = \gamma \left(1 + \frac{v}{c^2} v_x' \right)$$

$$v_x = \frac{dx}{dt} = \frac{v_x' + v}{1 + \frac{v v_x'}{c^2}}$$

$$v_y = \frac{dy}{dt} = \frac{dy'}{\gamma(dt' + \frac{v}{c^2} dx')}$$

4 vectors

$(t, x, y, z) = (x^0, x^1, x^2, x^3)$

invariant length $x_0^2 - \vec{x}^2$

four vector a set of quantities that transforms as (x^0, x^u)

$$A^0 = \gamma (A^{0'} + \frac{v}{c} A^{1'}) \quad v \parallel 1\text{-axis}$$

$$A^1 = \gamma (A^{1'} + \frac{v}{c} A^{0'})$$

$$A^2 = A^{2'}$$

$$A^3 = A^{3'}$$

$A^0^2 - \vec{A}^2 =$
invariant

metric tensor

$$g^{uv} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

$$g_{uv} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

$$g^{uv} g_{vu} = \delta^u_u$$

Greek indices 0, 1, 2, 3

Latin indices 1, 2, 3

repeated indices imply summation

contravariant vector $A^u = g^{uv} A_{v'}$

covariant vector $A_u = g_{uv} A^{v'}$

Scalar product

$$A_\mu B^\mu = A_0 B^0 + A_1 B^1 + A_2 B^2 + A_3 B^3$$

$$= A_0 B_0 - A_1 B_1 - A_2 B_2 - A_3 B_3$$

time like $A_\mu B^\mu > 0$

space like $A_\mu B^\mu < 0$

null vector $A_\mu B^\mu = 0$

notation $A^\mu = (A_0, \vec{A})$
 $x^\mu = (ct, \vec{r})$

tensor is defined by its transformation properties

$$A'^{\mu\nu} = L^{\mu\alpha} L^{\nu\beta} A^{\alpha\beta}$$

↑ Lorentz transformation.

- Contravariant $A^{\mu\nu}$
- Covariant $A_{\mu\nu}$
- mixed $A^\mu_\nu \quad A^\nu_\mu$

$$A^\mu_\nu = g_{\nu\alpha} A^{\mu\alpha}$$

$$A_{\mu\nu} = g_{\mu\alpha} g_{\nu\beta} A^{\alpha\beta}$$

Symmetric tensor $A_{\mu\nu} = A_{\nu\mu}$

Anti-symmetric tensor $A_{\mu\nu} = -A_{\nu\mu}$

Trace of tensor $A^a_{\mu} = A^0_0 + A^1_1 + A^2_2 + A^3_3$

$$\delta^{\nu}_{\mu} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$g_{\mu\nu} = g_{\mu\sigma} \delta^{\sigma}_{\nu}$$

$$g_{\mu\nu} = g^{\mu\nu}$$

Group theoretical analysis of Lorentz transformation

invariant $A_{\mu} A^{\mu} = A'_{\mu} A'^{\mu}$

$$\Rightarrow A'_{\mu} A'^{\mu} = A'^{\mu'} g_{\mu'm'} A'^{\mu}$$

$$= L^{\mu'}_{\mu''} A^{\mu''} g_{\mu'm'} L^{\mu}_{\mu'''} A^{\mu'''} = A^{\mu'} g_{\mu'm'} A^{\mu}$$

True for all A_{μ}

$$\Rightarrow g_{\mu'm'} = L^{\mu'}_{\mu''} g_{\mu'm'} L^{\mu}_{\mu''}$$

$$\Rightarrow \boxed{g = L^T g L}$$

$$\det L^T = \det L$$

$$\Rightarrow (\det L)^2 = 1$$

$$\Rightarrow \det L = \pm 1$$

proper Lorentz transformations are those that can continuously be transformed to the identity $\Rightarrow \det L = 1$

This cannot be done for improper Lorentz transformations

$$\begin{aligned} \det A = -1 &\iff \text{improper} \\ \det A = 1 &\iff \text{proper} \end{aligned}$$

reverse is not true, eg $A = -1$ then $\det A = 1$ but A is improper.

Lorentz group $L^T g L = g$

set of matrices that satisfy this relation is the Lorentz group.

$$g^T = g \Rightarrow \exists 10 \text{ independent relations}$$

L has 16 parameters \Rightarrow only 10 are independent.

Prove that it's a group

- identity satisfies this relation
- inverse satisfies the relation

$$L^T g L = g \Rightarrow L^{-1} = g L^T g$$

$$\Rightarrow (L^{-1})^T g L^{-1} = g L g g L^{-1} = g$$

- group property: if L_1 and L_2 are Lorentz then also $L_1 L_2$

$$L_1^T g L_1 = g \quad L_2^T g L_2 = g$$

$$\begin{aligned} \text{then } (L_1 L_2)^T g L_1 L_2 &= L_2^T L_1^T g L_1 L_2 \\ &= L_2^T g L_2 = g \end{aligned}$$

The Lorentz group is continuous group.

Therefore a large Lorentz transformation can be obtained by multiplying infinitesimal Lorentz transformations

$$L = 1 + \epsilon A \Rightarrow L_{\text{big}} = \left(1 + \frac{\epsilon}{N} A\right)^N = e^{\epsilon A}$$

\Rightarrow we only have to study the infinitesimal Lorentz transformations.

$$L^T g L = g \Rightarrow A^T g + g A = 0 \text{ to } O(\epsilon)$$

$$\Rightarrow A^T = -g A g$$

$$\Rightarrow (A^T)_{\alpha\beta} = -g_{\alpha\alpha} A_{\alpha\beta} g_{\beta\beta}$$

$$\stackrel{||}{=} A_{\beta\alpha} \Rightarrow A_{\alpha\alpha} = 0$$

$$A = \begin{pmatrix} 0 & A_{01} & A_{02} & A_{03} \\ A_{01} & 0 & A_{12} & A_{13} \\ A_{02} & -A_{12} & 0 & A_{23} \\ A_{03} & -A_{13} & -A_{23} & 0 \end{pmatrix}$$

\Rightarrow only 6 independent parameters

generators set of basis matrices

$$S_1 = \begin{pmatrix} \theta & \theta \\ \theta & i\theta \end{pmatrix} \quad S_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad S_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$K_1 = \begin{pmatrix} 0 & 1 & \theta \\ i\theta & 0 & \theta \\ \theta & \theta & 0 \end{pmatrix} \quad K_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ \theta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad K_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & \theta & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

commutation relations

$$[S_i, S_j] = \epsilon_{ijk} S_k \text{ (rotation group)}$$

$$[K_i, K_j] = -\epsilon_{ijk} K_k$$

$$[S_i, K_j] = \epsilon_{ijk} K_k$$

we write $A = -\vec{\omega} \cdot \vec{S} - \vec{g} \cdot \vec{K}$

Large Lorentz transformations

$$L = e^{-\vec{\omega} \cdot \vec{S} - \gamma_0 \vec{K}}$$

rotation about x axis

$$e^{-\omega_1 S_1} = 1 - \omega_1 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} + \frac{1}{2} \omega_1^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 - \frac{\omega_1^2}{2} & -\omega_1 \\ \omega_1 & \omega_1 & 1 - \frac{\omega_1^2}{2} \end{pmatrix} = \begin{pmatrix} \cos \omega_1 & -\sin \omega_1 & 0 \\ \sin \omega_1 & \cos \omega_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Special Lorentz transformation

$$e^{-\gamma_1 K_1} = 1 + \gamma_1 \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{2} \gamma_1^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 + \frac{1}{2} \gamma_1^2 & -\gamma_1 & 0 \\ -\gamma_1 & 1 + \frac{1}{2} \gamma_1^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cosh \gamma_1 & -\sinh \gamma_1 & 0 \\ -\sinh \gamma_1 & \cosh \gamma_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Invariant tensors

$$g'_{\mu\nu} = L_{\mu\alpha} L_{\nu\beta} g_{\alpha\beta} = (L g L^T)_{\mu\nu}$$

$$L^T g L = g \Rightarrow L^T = g^{-1} L^{-1} g = g L^{-1} g$$

$$\Rightarrow L g L^T = L g^2 L^{-1} g = g$$

$\Rightarrow g$ is invariant

$\Rightarrow \delta^{\mu\nu} = g^{\mu\alpha} g_{\alpha\nu}$ is invariant

A tensor that is symmetric in one frame is symmetric in all frames.

$$A'_{\mu\nu} = L_{\mu}^{\alpha} L_{\nu}^{\beta} A_{\alpha\beta}$$

$$\quad \quad \quad \parallel$$

$$\quad \quad \quad A_{\alpha\beta}$$

$$= L'_{\nu}{}^{\alpha} L_{\mu}^{\beta} A_{\alpha\beta} = A'_{\nu\mu}$$

same is $A_{\mu\nu} = -A_{\nu\mu}$ then $A'_{\mu\nu} = -A'_{\nu\mu}$

$A_{\mu\nu}$ is tensor of rank 2

$A_{\mu\nu\rho}$ is tensor of rank 3

then symmetry or anti-symmetry in any of the indices is preserved

completely anti-symmetric tensor

$A_{\mu\nu\rho}$ is completely antisymmetric under the interchange of any pair of indices.

This property is preserved only Lorentz transformations.

$A_{\mu\nu\rho\sigma} = 0$ if two or more indices are equal

$\Rightarrow A_{\mu\nu\rho\sigma}$ has only one independent component.

If $A_{123} = 1$ then this tensor is denoted by $\epsilon_{\mu\nu\rho\sigma}$

even

$\epsilon_{\mu\nu\rho\sigma} = 1$ if $\mu\nu\rho\sigma$ is a permutation of 0123

$= -1$ if $\mu\nu\rho\sigma$ is an odd permutation of 0123

$$\epsilon'_{m'p'q'r'} = L_{\mu'}^{\mu} L_{\nu'}^{\nu} L_{\rho'}^{\rho} L_{\sigma'}^{\sigma} \epsilon_{\mu\nu\rho\sigma}$$

$$\Rightarrow \epsilon'_{0123} = \det L$$

$\Rightarrow \epsilon$ is pseudotensor

Dual tensor $A^{*\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} A_{\rho\sigma}$
↑
Hodge star

$A^{*\mu\nu} A_{\mu\nu}$ is a pseudoscalar eg if $A_{\mu\nu} = x_{\mu} x_{\nu}$ then is always contains the time component

relations

$$\epsilon^{\mu\nu\rho\sigma} \epsilon_{\alpha\beta\rho\sigma} = -2 (\delta^{\mu}_{\alpha} \delta^{\nu}_{\beta} - \delta^{\mu}_{\beta} \delta^{\nu}_{\alpha})$$

$$\epsilon^{\mu\nu\rho\sigma} \epsilon_{\alpha\nu\rho\sigma} = -6 \delta^{\mu}_{\alpha}$$

$$\epsilon^{\mu\nu\rho\sigma} \epsilon_{\mu\nu\rho\sigma} = 24$$

integration

4 volume element

$$d\Omega = dx^0 dx^1 dx^2 dx^3$$

$d\Omega$ is Lorentz invariant

Gauss theorem $\int A^\mu \frac{1}{6} \epsilon^{\mu\nu\rho\sigma} dS_{\nu\rho\sigma}$
↖ 3 surface element

$$= \int \partial_\mu A^\mu d\Omega$$

Stokes theorem

$$\frac{1}{2} \int A^{\mu\nu} \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} dF^{\rho\sigma} = \int dS_\mu \cdot \frac{\partial A^{\mu\nu}}{\partial x^\nu}$$

↙ 2 surface element
||
↘ $\frac{1}{6} \epsilon_{\mu\nu\rho\sigma} dS^{\nu\rho\sigma}$

$$\oint A_\mu dx^\mu = \frac{1}{2} \int dF^{\mu\nu} (\partial_\nu A_\mu - \partial_\mu A_\nu)$$

Four velocity

$$u^{\mu} = \frac{dx^{\mu}}{ds}$$

$$ds = c dt \sqrt{1 - \beta^2}$$

$$u^k = \frac{dx^k}{dt} \frac{dt}{ds} = \gamma \frac{v^k}{c}$$

$$u^0 = \frac{dx^0}{ds} = \gamma$$

$$\Rightarrow u = \begin{pmatrix} \gamma \\ \gamma \frac{\vec{v}}{c} \end{pmatrix}$$

Relativistic action of a free particle

Euler Lagrangian equations

$$S = \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t) dt$$

↑ generalized coordinates

generalized momenta $p_i = \frac{\delta L}{\delta \dot{q}_i}$

eq of motion $\delta S = 0$

$$\int_{t_1}^{t_2} \left(\frac{\delta L}{\delta q_i} \delta q_i + \frac{\delta L}{\delta \dot{q}_i} \delta \dot{q}_i \right) dt$$

↑ partial integrate

$$\Rightarrow \frac{\delta L}{\delta q_i} - \frac{d}{dt} \frac{\delta L}{\delta \dot{q}_i} = 0$$

In relativity, the \mathcal{L} equation (148) have to be scalar. Otherwise the eqs of motion would be different in different frames.

Only invariant for a free particle

$$ds$$

$$\Rightarrow S = \alpha \int_a^b ds$$

We determine α from the nonrelativistic limit

$$S = \alpha \int_a^b \frac{ds}{dt} dt = \alpha \int_{t_1}^{t_2} \sqrt{c^2 - v^2} dt$$

$$\Rightarrow \text{Lagrangian density: } L = \alpha \sqrt{c^2 - v^2} \\ = \alpha c \left(1 - \frac{1}{2} \frac{v^2}{c^2}\right)$$

$$\text{NR: } L = T - V \Rightarrow$$

$$= \frac{1}{2} m v^2 - V = \alpha c - \alpha c \frac{1}{2} \frac{v^2}{c^2}$$

$$\Rightarrow \alpha = -m c \quad \Rightarrow V = m c^2$$

$$\Rightarrow L = -\frac{m c^2}{\gamma}$$

Momentum and force

generalized coordinates x^u

generalized velocities $u^u = \frac{dx^u}{ds}$

generalized momenta $P_u = - \frac{\delta S}{\delta x^u}$

$$\begin{aligned} &= - \frac{\delta}{\delta x^u} \left(-mc \int_a^b \sqrt{dx^u dx^u} \right) \\ \delta S &= -2m \int_a^b \frac{1}{2} \frac{dx^u \delta dx^u}{ds} \\ &= -mc \int_a^b u^u \delta dx^u \\ &= -mc u^u \delta x^u \Big|_a^b + mc \int_a^b \frac{du^u}{ds} \delta x^u ds \end{aligned}$$

endpoints fixed: $\delta x^u|_a = \delta x^u|_b = 0$

$\delta S = 0$ for arbitrary variation

$$\Rightarrow \frac{du^u}{ds} = 0$$

To calculate the momentum keep $x^u|_a$ fixed but keep $x^u|_b$ variable.

$$P_u = - \frac{\delta S}{\delta x^u|_b}$$

$$\begin{aligned} S &= -mc \int_a^b \sqrt{dx^u dx^u} = -mc \int_a^b \frac{ds}{dx^u} dx^u \\ &= -mc \int_a^b \frac{dx^u}{ds} dx^u = -mc \int_a^b u^u dx^u \end{aligned}$$

$$\Rightarrow p_a = - \frac{\delta S}{\delta x^a_b} = + m c u_a$$

$$p^0 = m c \frac{c}{c} \gamma = \gamma^0 m$$

$$p_i = m c \gamma = \frac{E}{c}$$

$(\frac{E}{c}, \vec{p})$ is a 4 vector.

$(\frac{E}{c})^2 - \vec{p}^2$ is invariant

for $\vec{v} = 0$ it is $m^2 c^2$
 $\Rightarrow (\frac{E}{c})^2 - \vec{p}^2 = m^2 c^2$

Note that
$$S = -m c \int \sqrt{dx^\mu dx_\mu}$$
$$= -m c \int \sqrt{\frac{dx^\mu}{ds} \frac{dx_\mu}{ds}} ds$$
$$= -m c \int \sqrt{u^\mu u_\mu} ds$$

$$u_\mu u^\mu = 1$$

but variations are over trajectories for which $u_\mu u^\mu \neq 1$

- Four momentum is conserved in absence of external forces
- four force
$$g^a = \frac{dp^a}{ds} = m c \frac{du^a}{ds}$$

Relativistic Kinematics

(129)

$$p_n^{\text{initial}} = p_n^{\text{final}}$$

$$p_n p^{\mu} = m^2 c^2 \text{ for each particle}$$

Four currents

$$\text{continuity eq. } \partial_t \rho + \vec{\nabla} \cdot \vec{j} = 0$$

$$\Rightarrow \partial_{ct} (\rho) + \partial_{x^{\mu}} j^{\mu} = 0$$

$$\Rightarrow \partial_{x^{\mu}} j^{\mu} = 0$$

$$j^{\mu} = (\rho, \vec{j})$$

must be invariant bc charge is conserved

$\Rightarrow j^{\mu}$ is a 4-current

Covariant gauge

$$\vec{\nabla} \cdot \vec{A} + \frac{1}{c} \partial_t \phi = 0$$

$$\Rightarrow \begin{aligned} (\vec{\nabla}^2 - \partial_t^2) \phi &= -4\pi\rho \\ (\vec{\nabla}^2 - \partial_t^2) \vec{A} &= -\frac{4\pi}{c} \vec{j} \end{aligned}$$

$\Rightarrow (\phi, \vec{A})$ is a 4 vector

Lagrangian for An free

we have 4 vectors u_μ, A^μ
nontrivial scalar $u_\mu A^\mu$

Invariant Lagrangian

$$L = -\frac{mc^2}{\gamma} + \frac{a}{\gamma} u_\mu A^\mu$$

$$S = \int L dt$$

$$ds^2 = dx^{\mu} dx_{\mu} = dx^2 - d\vec{x}^2$$

$$\Rightarrow \frac{ds}{d\tau} = \sqrt{c^2 - v^2}$$

$$\Rightarrow ds = c \gamma^{-1} dt$$

$\gamma^{-1} dt$ is invariant

NR limit

$$\frac{a u_0 A_0}{\gamma} = \frac{a c \phi}{\gamma} = -V$$

↑
potential energy
 $V = e \phi$

$$\Rightarrow a = -\frac{e}{c}$$

$$\Rightarrow L = -\frac{mc^2}{\gamma} - \frac{e}{c} \frac{u_\mu A^\mu}{\gamma} \quad u_\mu = (\gamma c, \gamma \vec{u})$$

Euler Lagrange equations

$$-\frac{d}{dt} \frac{\delta L}{\delta \dot{u}_i} + \frac{\delta L}{\delta q_i} = 0 \quad u_i = \dot{q}_i$$

$$\Rightarrow -\frac{d}{dt} \left(+m \dot{q}_i - \frac{e}{c} A^i \right) - e \frac{\delta \phi}{\delta q_i} - \frac{e}{c} u_\mu \frac{\delta A^\mu}{\delta q_i} = 0$$

$$\Rightarrow \frac{d p_i}{dt} = + \frac{e}{c} \frac{\partial A^i}{\partial x_\mu} \dot{q}_\mu - \frac{e}{c} u_\mu \frac{\partial A^\mu}{\partial q_i} - e \frac{\delta \phi}{\delta q_i}$$

$$u^\mu = \begin{pmatrix} c\gamma \\ \vec{v}\gamma \end{pmatrix}$$

$$\Rightarrow \frac{d \vec{p}}{dt} = +e \vec{E} + \frac{e}{c} \vec{u} \times \vec{B}$$

\vec{u} with upper index

Lorentz force

Canonical momentum

$$p_\mu = -\frac{\delta L}{\delta \dot{u}^\mu} = \underbrace{m \dot{u}_\mu}_{\text{kinematical momentum}} + \frac{e}{c} A_\mu$$

kinematical momentum

Electromagnetic field tensor

$$\vec{E} = -\frac{1}{c} \partial_t \vec{A} - \vec{\nabla} \phi$$

$$E_k = -\frac{\partial A_k}{\partial x_0} - \partial_{x_k} A^0$$

$$= -\partial^0 A_k + \partial^k A^0$$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

$$B_j = \partial_1 A_2 - \partial_2 A_1$$

$$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha$$

↑ em field tensor

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

dual field tensor

$${}^*F_{\alpha\beta} = \frac{1}{2} \epsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix}$$

$$\begin{aligned} \vec{E} &\rightarrow \vec{B} \\ \vec{B} &\rightarrow -\vec{E} \end{aligned}$$

Transformation of \vec{E} and \vec{B}

$$F_{\alpha\beta} \text{ in } K$$

$$F'_{\alpha\beta} \text{ in } K'$$

$$F'_{\alpha\beta} = L^\alpha_\gamma L^\beta_\delta F^{\gamma\delta}$$

$$F' = L F L^T$$

boost along 1 axis

$$L = \begin{pmatrix} \gamma - \beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow E'_1 = E_1$$

$$B'_1 = B_1$$

$$E'_2 = \gamma(E_2 - \beta E_3)$$

$$B'_2 = \gamma(B_2 + \beta E_3)$$

$$E'_3 = \gamma(E_3 + \beta E_2)$$

$$B'_3 = \gamma(B_3 - \beta E_2)$$

General form

$$\vec{E}' = \gamma(\vec{E} + \vec{\beta} \times \vec{B}) - \frac{\gamma^2}{1+\gamma} \vec{\beta} (\vec{\beta} \cdot \vec{E})$$

$$\vec{B}' = \gamma(\vec{B} - \vec{\beta} \times \vec{E}) - \frac{\gamma^2}{1+\gamma} \vec{\beta} (\vec{\beta} \cdot \vec{B})$$

- \vec{E} and \vec{B} are mixed by Lorentz transformations

Covariant form of ∇F

tensors ∂_α , J^α , $F^{\alpha\beta}$, $\tilde{F}^{\alpha\beta}$

$$\partial_\alpha F^{\alpha\beta} \sim J^\beta$$

only possibility
by tensor structure
and dimensionality.

$$\partial_\alpha F^{\alpha\beta} = \frac{4\pi}{c} J^\beta$$

$$\rho = 0 \quad \vec{\nabla} \cdot \vec{E} = 4\pi\rho$$

$$\beta = 1, 4, 0 \quad \vec{\nabla} \times \vec{B} - \frac{1}{c} \partial_t \vec{E} = \frac{4\pi}{c} \vec{j}$$

Other covariant equation

$$\partial_\alpha \tilde{F}^{\alpha\beta} = 0$$

$$A = 0 \quad \text{div } \vec{B} = 0$$

$$\beta = 1, 2, 3 \quad \vec{\nabla} \times \vec{E} + \frac{1}{c} \partial_t \vec{B} = 0$$

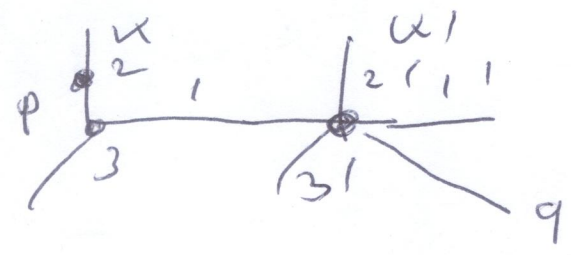
Bianchi identity

in terms of $F^{\alpha\beta}$

$$\partial^\alpha F^{\beta\gamma} + \partial^\beta F^{\gamma\alpha} + \partial^\gamma F^{\alpha\beta} = 0$$

Example

charge q at o in K'



$$E_1' = -\frac{qvt'}{r'^3} \quad E_2' = \frac{qb}{r'^3} \quad E_3' = 0$$

$$r' = \sqrt{b^2 + v^2 t'^2}$$

Field in K

$$E_1 = E_1' = -\frac{qvt}{(\sqrt{b^2 + \gamma^2 v^2 t^2})^3}$$

$$t' = \gamma(t - \frac{v}{c} x_1)$$

" at P

$$\Rightarrow t' = \gamma t$$

$$E_2 = \gamma E_2' = \frac{\gamma qb}{(\sqrt{b^2 + \gamma^2 v^2 t^2})^3}$$

$$B_3 = \gamma \beta E_2' = \beta E_2$$

$$B \rightarrow 0 \quad \vec{B} = \frac{q}{c} \frac{\vec{v} \times \vec{r}}{r^3}$$