Integral form of Gauss law

$$\int_V dV\, \vec{\nabla}\cdot\vec{E} = \int dV\, 4\pi\rho$$

$$\parallel$$

$$\int_{\partial V} da\, \vec{n}\cdot\vec{E}$$

e) Poisson law

$$\left.\begin{array}{l}\vec{\nabla}\vec{E} = 4\pi\rho \\ \vec{E} = -\vec{\nabla}\phi\end{array}\right\} \Rightarrow \vec{\nabla}^2\phi = -4\pi\rho$$

Poisson equatia

$\rho = 0 \quad \vec{\nabla}^2\phi = 0$ Laplace eq

solutions are called harmonic function.

f) Uniqueness

A vector field that satisfies

$\vec{\nabla}\vec{E} = 0 \qquad \vec{\nabla}\times\vec{E}$

and vanishes at least as $\frac{1}{r^2}$ for $r\to\infty$

vanishes everywhere

This follows from liouville's theoreme:

if f is a harmonic function on $\mathbb{R}^n$

and f is bounded from above or below

then f is constant

Reason $\vec{E} = -\vec{\nabla}\phi$

$$\vec{\nabla}^2 \phi = 0$$

$\vec{E} \sim \frac{1}{r^2}$ for $r \to \infty \Rightarrow \phi \sim \frac{1}{r}$ for $r \to \infty$

$\vec{\nabla} \cdot \vec{E} = 0 \Rightarrow \exists$ no singularities

$\Rightarrow \phi$ is bounded $\Rightarrow \phi =$ constant

$\Rightarrow \vec{E} = 0$

Consequence: **uniqueness theorem**

$$\left.\begin{array}{l} \vec{\nabla} \cdot \vec{E}_1 = 4\pi\rho_1 \\ \vec{\nabla} \cdot \vec{E}_2 = 4\pi\rho_2 \end{array}\right\} \Rightarrow \vec{\nabla}(\vec{E}_1 - E_2) = 0$$

statics $\Rightarrow \vec{\nabla} \times (\vec{E}_1 - \vec{E}_2) = 0$

$\Rightarrow E_1 - E_2 = 0$

Solutions of the Laplace eq

$\vec{\nabla}^2 \phi = 0$

$z = r\cos\theta$
$x = r\sin\theta\cos\varphi$
$y = r\sin\theta\sin\varphi$

spherical coordinates

$$\vec{\nabla}^2 \phi = \frac{1}{r}\partial_r^2 (r\phi) + \frac{1}{r^2\sin\theta}\partial_\theta\left(\sin\theta \frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2}$$

$$\phi = \frac{u(r)}{r} P(\theta) Q(\varphi)$$

separation of variables

$$\nabla^2\phi = \frac{PQ}{r}\partial_r^2 u + \frac{uQ}{rr^2\sin\theta}\partial_\theta(\sin\theta\,\partial_\theta P)\,\underline{g}$$

$$+ \frac{uP}{rr^2\sin^2\theta}\frac{\partial^2 Q}{\partial\varphi^2} = 0$$

divide by uPQ

$$\frac{u^{-1}}{r}\partial_r^2 u + \frac{P^{-1}}{rr^2\sin\theta}\partial_\theta(\sin\theta\,\partial_\theta P)$$

$$+ \frac{Q^{-1}}{rr^2\sin^2\theta}\frac{\partial^2 Q}{\partial\varphi^2} = 0$$

$\times r^3$

$$r^2 u^{-1}\partial_r^2 u + \frac{P^{-1}}{\sin^2\theta}\partial_\theta(\sin\theta\,\partial_\theta P)$$

$$+ \frac{Q^{-1}}{\sin^2\theta}\frac{\partial^2 Q}{\partial\varphi^2} = 0$$

↑ only depends on r ↗ only depends on θ, φ

$\Rightarrow$ both should be constant because the equation is valid for all r, θ, φ

$r^2 u^{-1}\partial_r^2 u = c \Rightarrow \partial_r^2 u = c\frac{u}{r^2}$

$u = \alpha r^{l+1} \Rightarrow l(l+1)r^{l-2} = c\,r^{l-2}$

$\Rightarrow c = l(l+1)$

then

$$\ell(\ell+1) \sin^2\theta + P^{-1}\partial_\theta(\sin\theta\,\partial_\theta P)$$

$$+ Q^{-1}\frac{\partial Q^2}{\partial\varphi^2} = 0$$

$$\Rightarrow \ell(\ell+1)\sin^2\theta + P^{-1}\partial_\theta(\sin\theta\,\partial_\theta P)$$

$$= -\left(\frac{\partial Q^2}{\partial \varphi^2}\right) Q^{-1}$$

↑ function of θ ↗ function of φ

equal $\forall\, \theta, \varphi \Rightarrow$ they have to be constant

$$\Rightarrow Q^{-1}\frac{\partial^2 Q^2}{\partial\varphi^2} = -m^2 \Rightarrow Q = e^{\pm i m \varphi}$$

$$\Rightarrow -\partial_\theta\,(\sin\theta\,\partial_\theta P) = \ell(\ell+1)\sin^2\theta\, P - m^2 P$$

This equation is known as the Legendre equation.

The solutions are given by the associated Legendre polynomials

The combination PQ is given by $PQ = c' P_{\ell m}(\cos\theta)\, e^{im\varphi} \equiv c' Y_{\ell m}(\theta,\varphi)$

↑ spherical harmonics

The constant c' is fixed by the normalization sum rule

$$\sum_{m=-\ell}^{\ell} Y_{\ell m}(\theta,\varphi)\, Y^*_{\ell m}(\theta,\varphi) = \frac{2\ell+1}{4\pi}$$

$$Y_{00} = \frac{1}{\sqrt{4\pi}}$$

$$Y_{11} = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi} = -\sqrt{\frac{3}{8\pi}}\,\frac{x+iy}{r}$$

$$Y_{10} = \sqrt{\frac{3}{4\pi}}\cos\theta = \sqrt{\frac{3}{4\pi}}\,\frac{z}{r}$$

$$Y_{1-1} = \sqrt{\frac{3}{8\pi}}\sin\theta\, e^{-i\varphi} = \sqrt{\frac{3}{8\pi}}\,\frac{x-iy}{r}$$

We have found an infinite set of harmonic functions

$$\nabla^2 r^{\ell}\, Y_{\ell m}(\theta,\varphi) = 0$$

$$\ell(\ell+1) = (-\ell-1)((-\ell-1)+1)$$

$$\Rightarrow u(r) = \frac{1}{r^{\ell}} \text{ is also solution}$$

$$\Rightarrow \quad \vec{\nabla}^2 \frac{1}{r^{\ell+1}} Y_{\ell m}(\theta, \varphi) = 0$$

Most general solution of the Laplace eq.

$$\phi(r, \theta, \varphi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left(A_{\ell m} r^{\ell} + B_{\ell m} r^{-\ell-1} \right) Y_{\ell m}(\theta, \varphi)$$

This is a complete set of functions

Strategy for solution of Poisson eq

$$\nabla^2 \phi = -4\pi \rho$$

then $\nabla^2(\phi + \phi_h) = -4\pi \rho$

ϕ ↗ special solution

$$\vec{\nabla}^2 \phi_h = 0$$

Generally we have a charge distribution and boundary conditions

The ϕ_h are determined by the boundary conditions.

outside charge distribution ; $\rho = 0$

$\Rightarrow \vec{\nabla}^2 \phi = 0$ $\quad \phi \to 0$ for $r \to \infty \Rightarrow A_{\ell m} = 0$

inside charge distribution : solution should be regular $\Rightarrow B_{\ell m} = 0$

Some properties of spherical harmonics

$$Y_{\ell m}(\theta,\varphi) = \sqrt{\frac{2\ell+1}{4\pi}}\sqrt{\frac{(\ell-m)!}{(\ell+m)!}}\, P_{\ell m}(\cos\theta)\, e^{im\varphi}$$

$$r^2 \vec{\nabla}^2 Y_{\ell m}(\theta,\varphi) = -\ell(\ell+1) Y_{\ell m}(\theta,\varphi)$$

$$Y_{\ell -m}(\theta,\varphi) = (-1)^m Y^*_{\ell m}(\theta,\varphi)$$

orthogonality

$$\int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta\, d\theta\, Y^*_{\ell' m'}(\theta,\varphi) Y_{\ell m}(\theta,\varphi) = \delta_{\ell\ell'}\delta_{mm'}$$

completeness

$$\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Y^*_{\ell m}(\theta',\varphi') Y_{\ell m}(\theta,\varphi) = \delta(\varphi-\varphi') \times \delta(\cos\theta - \cos\theta')$$

$\Rightarrow$ An arbitrary function of θ and φ can be expressed as

$$f(\theta,\varphi) = \sum a_{\ell m} Y_{\ell m}(\theta,\varphi)$$

sum rule

$$\sum_{m=-\ell} Y_{\ell m}(\theta,\varphi) Y^*_{\ell m}(\theta,\varphi) = \frac{2\ell+1}{4\pi}$$

<u>Example 1</u>: single charge at 0

$$\rho(r) = q\,\delta^3(r)$$

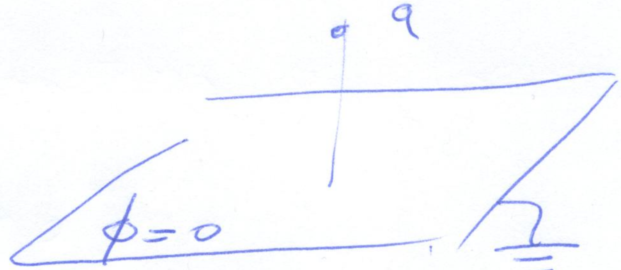
potential is spherically symmetric

$\Rightarrow$ only $\ell = 0$, $m = 0$ is allowed

$$\Rightarrow \quad \phi = A_{00} + B_{00}\, r^{-1}$$

$\phi \to 0$ for $r \to \infty$ $\Rightarrow A_{00} = 0$

<u>Example 2</u>



solution is axial symmetric

$\Rightarrow m = 0$ $\qquad$ $\phi \to 0$ for $r \to \infty$

$\Rightarrow A_{\ell m} = 0$ for $r > 0$

$$\Rightarrow \quad \phi = \sum B_{\ell m}\, r^{-\ell-1}\, Y_{\ell 0}(\theta, \varphi)$$

does not depend on φ.

$z = 0$ $\quad \theta = \frac{\pi}{2}$

$$\sum B_{\ell m}\, r^{-\ell-1}\, Y_{\ell 0}\left(\tfrac{\pi}{2}, \varphi\right) = 0$$

only odd ℓ are possible

$$Y_{2k+1,0}\left(\tfrac{\pi}{2}, \varphi\right) = 0$$

$$\Rightarrow \phi = \sum_{\ell\, odd} a_\ell\, r^{-\ell-1}\, Y_{(\ell,0)}(\theta, \varphi) \quad \left(Y_{\ell 0}(0) = \sqrt{\frac{2\ell+1}{4\pi}}\right)$$

look at the z-axis $\theta_0 = 0$

$$\Rightarrow \phi = \sum_{\ell\, odd} b_\ell\, r^{-\ell-1}$$

$$= \sum_\ell \frac{(-(-1)^\ell + 1)}{2} b_\ell\, r^{-\ell-1}$$

for

should behaves as $\frac{q}{|z-a|}$ for $z \to a$

$a_\ell = \sqrt{\frac{4\pi}{2\ell+1}}\, b_\ell$

$$\Rightarrow b_\ell = q a^\ell\, 2 \quad \text{geometric series}$$

on z axis $\phi = \frac{q}{(r-a)} - \frac{q}{r+a}$

$$\Rightarrow \phi = \sum (1 - (-1)^\ell) \frac{q a^\ell}{r^{\ell+1}} Y_{\ell 0}(\theta, \varphi) \frac{\sqrt{\pi 4}}{\sqrt{2\ell+1}}$$

potential on z-axis

$$\phi = \sum_{\ell=0}^{\infty} \left((1) - (-1)^{\ell} \right) \frac{q a^{\ell}}{r^{\ell+1}} \underbrace{Y_{\ell 0}(0,\varphi) \sqrt{\frac{4\pi}{2\ell+1}}}_{=1}$$

$$= \frac{q}{r} \frac{1}{1-\frac{a}{r}} - \frac{q}{r} \frac{1}{1+\frac{a}{r}}$$

$$= \frac{q}{r-a} - \frac{q}{r+a}$$

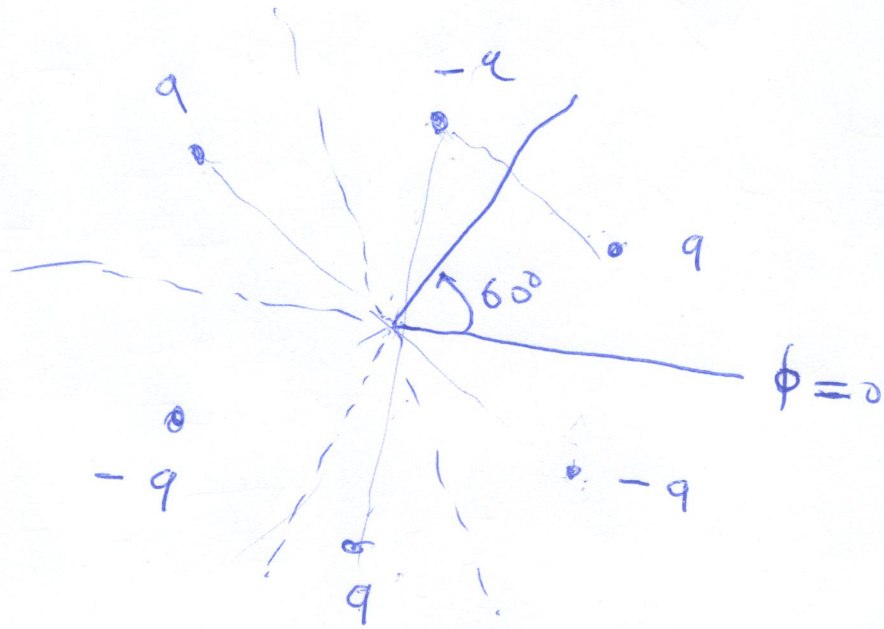
↑ image charge

This problem can be solved by image charges.
The trick is to find other charges that give the same boundary condition

$\Rightarrow$ Solution on the right of the plane is the same, not on the left

$$\phi(\vec{r}) = \frac{q}{|\vec{r}-\vec{a}|} - \frac{q}{|\vec{r}+\vec{a}|}$$

2nd Example of image charges



Expansion of $\frac{1}{|\vec{r}-\vec{r}'|}$ in spherical harmonics

choose $|\vec{r}|>|r'|$

$$\frac{1}{|\vec{r}-\vec{r}'|} = \sum \frac{1}{r^{\ell+1}} A_{\ell m} P_\ell Y_{\ell m}(\theta,\varphi)$$

choose $\vec{r}'$ on z axis

on z axis: $\frac{1}{|r-r'|} = \sum_{k=0}^{\infty} \frac{r'^k}{r^{k+1}}$

no dependence on $\varphi \Rightarrow m=0$

$$\Rightarrow A_{\ell 0} = \frac{r'^\ell}{Y_{\ell 0}(0,\varphi)} = r'^\ell \left(\frac{4\pi}{2\ell+1}\right)^{1/2} Y_{\ell 0}(0,\varphi)$$

$$\frac{1}{|\vec{r}-\vec{r}'|} = \sum \frac{1}{r^{\ell+1}} r'^\ell \frac{4\pi}{2\ell+1} Y_{\ell 0}(0,\varphi) Y_{\ell 0}(\theta,\varphi)$$

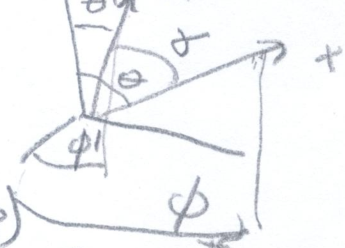
$$Y_{\ell 0}(\theta',\varphi) Y_{\ell 0}(\theta,\varphi)$$

only depends on angle between $\vec{r}$ and $\vec{r}'$

$$Y_{\ell 0}(\theta,\varphi) = \sqrt{\frac{2\ell+1}{4\pi}} P_\ell(\cos\theta)$$

$$Y_{\ell 0}(0,0) = \sqrt{\frac{2\ell+1}{4\pi}}$$

addition theorem

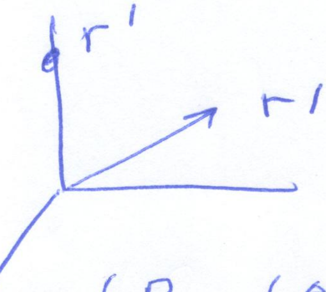


$$P_\ell(\cos\gamma) = \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{\ell} Y^*_{\ell m}(\theta',\varphi') \times Y_{\ell m}(\theta,\varphi)$$

$$\Rightarrow \frac{1}{|\vec{r}-\vec{r}'|} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{4\pi}{(2\ell+1)} \frac{r'^\ell}{r^{\ell+1}} Y^*_{\ell m}(\theta',\varphi') Y_{\ell m}(\theta,\varphi)$$

$$\frac{1}{|\vec{r}-\vec{r}'|} = \sum_\ell \frac{r'^\ell}{r^{\ell+1}} \sqrt{\frac{4\pi}{2\ell+1}}\, Y_{\ell 0}(\theta, \varphi)$$

now rotate $r' \to (\theta', \varphi')$



then $Y_{\ell m}(\vec{r}') = \sum_{m''} \mathcal{D}^\ell_{m'' m}\, Y_{\ell m''}(R_{\theta'\varphi'}(\theta, \varphi))$

$Y_{\ell m}$: $m \downarrow 0$; $\mathcal{D}^\ell_{m''m}$: $m \downarrow 0$

$\uparrow$ Wigner $\mathcal{D}$ matrices

$R_{\theta'\varphi'}(\theta,\varphi)$: new θ, φ coordinates of $\vec{r}$

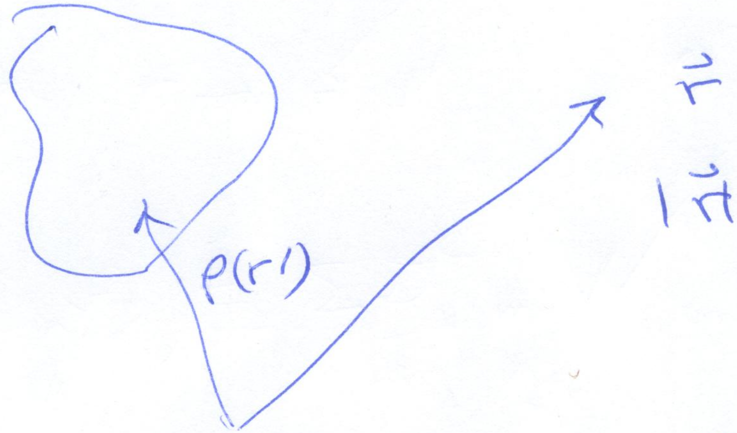
$$\mathcal{D}^\ell_{m''0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y^*_{\ell m''}(\theta', \varphi')$$

$$\Rightarrow \frac{1}{|\vec{r}-\vec{r}'|} = \sum_{\ell, m''} \frac{r'^\ell}{r^{\ell+1}} \frac{4\pi}{2\ell+1} Y^*_{\ell m''}(\theta', \varphi') Y_{\ell m''}(\theta, \varphi)$$

note that $r > r'$

Multipole expansion

a)



$|\vec{r}| > |r'|$

$$\phi(r) = \int d^3r' \, \rho(r') \frac{1}{|\vec{r}' - \vec{r}|}$$

$$= \int d^3r' \, \rho(r') \sum_{lm} \frac{r'^l}{r^{l+1}} \frac{4\pi}{2l+1} \rho(r') Y^*_{lm}(\theta', \varphi') \times Y_{lm}(\theta, \varphi)$$

multipole moment

$$Q_{lm} = \int d^3r' \, r'^l \rho(\vec{r}') \, Y^*_{lm}(\theta' \varphi')$$

$$\Rightarrow \phi(r) = \sum_{l=0}^{\infty} \frac{4\pi}{2l+1} \frac{1}{r^{l+1}} \sum_{m=-l}^{l} Y_{lm}(\theta, \varphi) \, Q_{lm}$$

$l=0$ monopole

$l=1$ dipole

$l=2$ quadrapole

b) <u>Dipole moment</u>

$$\vec{p} = \int d^3x' \, \rho(x') \, \vec{x}'$$

$$Q_{10} = \sqrt{\frac{3}{4\pi}} \int z' \rho(x') \, d^3x' = \sqrt{\frac{3}{4\pi}} \, p_z$$

$$Q_{11} = -\sqrt{\frac{3}{8\pi}} \int (x' - iy') \, \rho(x') \, d^3x'$$

$$= -\sqrt{\frac{3}{8\pi}} (p_x - i p_y)$$

$$Q_{1-1} = - Q_{11}^{*} = \sqrt{\frac{3}{8\pi}} (p_x + i p_y)$$

The potential of a dipole is given by

$$\phi(r) = \frac{4\pi}{3} \frac{1}{r^2} \left(Y_{11} \left(-\sqrt{\frac{3}{8\pi}}\right) (p_x - i p_y) \right.$$

$$+ Y_{10} \sqrt{\frac{3}{4\pi}} \, p_z$$

$$\left. + Y_{1-1} \sqrt{\frac{3}{8\pi}} (p_x + i p_y) \right)$$

↑ $2\ell + 1$

$$Y_{11} = -\sqrt{\frac{3}{8\pi}} (x + iy)/r \qquad Y_{10} = \sqrt{\frac{3}{4\pi}} \frac{z}{r}$$

$$Y_{1-1} = \sqrt{\frac{3}{8\pi}} \frac{(x - iy)}{r}$$

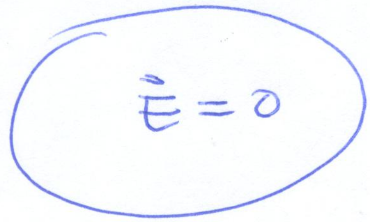
$$\phi(r) = \frac{4\pi}{3} \frac{1}{r^3} \left((x + iy)(p_x - i p_y) \frac{3}{8\pi} + z \, p_z \frac{3}{4\pi} \right.$$

$$\left. + (x - iy)(p_x + i p_y) \frac{3}{8\pi} \right)$$

$$= \frac{1}{r^3} (x p_x + y p_y + z p_z) = \frac{1}{r^3} (\vec{r} \cdot \vec{p})$$

Electrostatics of conductors

a)



$\vec{E} = 0 \Rightarrow \vec{\nabla}\vec{E} = 0 \Rightarrow \rho = 0$

$\Rightarrow$ we can only have a surface charge density on a conductor.

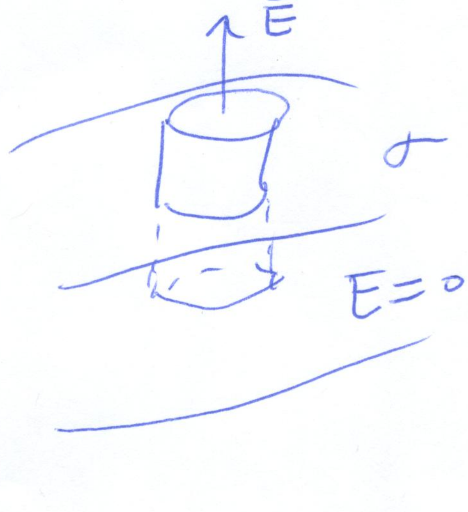
$\vec{E} = 0 \quad \vec{E} = \vec{\nabla}\phi \Rightarrow$

- conductor is an equipotential surface

$$\phi_{12} = -\int_1^2 \vec{E} \cdot ds$$

-  $\vec{E} \perp$ surface, otherwise the electrons will rearrange

- Surface charge density



$$\oint \vec{E} \cdot \vec{n}\, da = 4\pi\sigma A$$

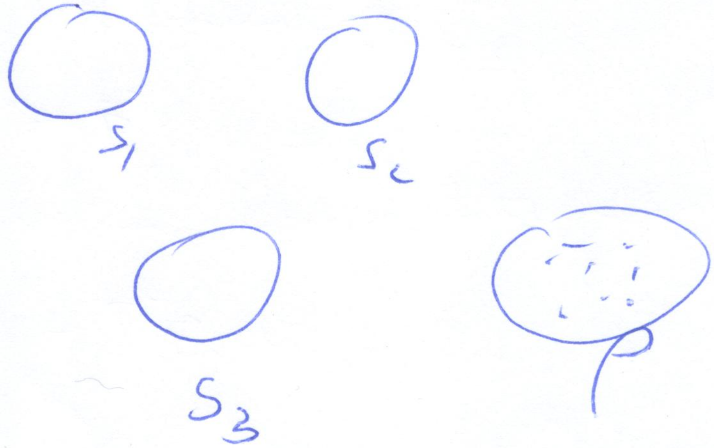
$$\Rightarrow E_\perp A = 4\pi\sigma A$$

$$E_\perp = 4\pi\sigma$$

Uniqueness theorem

give ρ, conducting surfaces S_i with charge Q_i or potential ϕ_i, then the electric field is determined uniquely

Proof



Suppose we have two different potentials

$$\vec{E}_1 = -\vec{\nabla}\psi_1 \qquad E_2 = -\vec{\nabla}\psi_2$$

then $I = \int_{V = \mathbb{R}^3 / \cup(S_i)} d^3r \, (\vec{\nabla}\psi_1 - \vec{\nabla}\psi_2)^2 \neq 0$

$$= \int_V d^3r \; \vec{\nabla}(\psi_1 - \psi_2)\,\vec{\nabla}(\psi_1 - \psi_2)$$

$$= \sum_i \int_{S_i} da \; \vec{n}\,(\psi_1^i - \psi_2^i)\,\vec{\nabla}(\psi_1^i - \psi_2^i) - \int_V d^3r (\psi_1 - \psi_2) \times \vec{\nabla}^2(\psi_1 - \psi_2)$$

ψ_k^i is constant on S_i; $\vec{\nabla}^2(\psi_1 - \psi_2) = 0$, since $\vec{\nabla}^2\psi_1 = \rho$, $\vec{\nabla}^2\psi_2 = \rho$

$$= \sum_i \int_{S_i} (\psi_1^i - \psi_2^i)\,\vec{n}\cdot(\vec{E}_{1i} - \vec{E}_{2i})$$

$$= \sum_i (\psi_1^i - \psi_2^i)(Q_1^i - Q_2^i) \Rightarrow \text{qed}$$

Green's function

Definition $\vec{\nabla}^2_{x'} G(\vec{x}, \vec{x}') = -4\pi\,\delta(\vec{x} - x')$

So $G(\vec{x}, x') = \dfrac{1}{|\vec{x} - \vec{x}'|} + F(\vec{x}, \vec{x}')$

with $\vec{\nabla}^2_{x'} F(\vec{x}, \vec{x}') = 0$

$\vec{F}$ is determined by the boundary conditions

for a single charge in vacuum

$$G(\vec{x}, \vec{x}') = \frac{1}{|\vec{x} - x'|}$$

↗ is potential at x from unit charge at x'

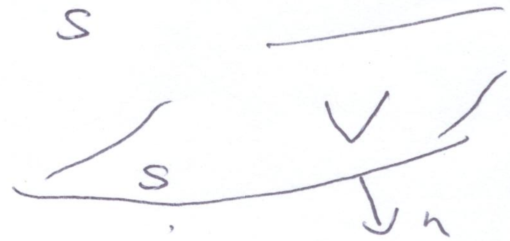
Green's function is symmetric

$$\int d^3r\; G(\vec{r}, \vec{r}_1)\, \underbrace{\vec{\nabla}^2_r G(\vec{r}, r_2)}_{-4\pi\,\delta(\vec{r} - \vec{r}_2)} = \int d^3r\; \underbrace{\vec{\nabla}^2_r G(\vec{r}, \vec{r}_1)}_{-4\pi\,\delta(\vec{r} - r_1)}\, G(\vec{r}, \vec{r}_2)$$

$$\Rightarrow -4\pi\, G(\vec{r}_2, \vec{r}_1) = -4\pi\, G(\vec{r}_1, \vec{r}_2)$$

Application of Green's Function

Gauss: $$\int_V \vec{\nabla}\cdot\vec{A}\, d^3x = \oint_S \vec{A}\cdot\vec{n}\, da$$



choose $\vec{A} = \phi\,\vec{\nabla}\psi$ ← arbitrary

Then $$\vec{\nabla}\vec{A} = \phi\,\vec{\nabla}^2\psi + \vec{\nabla}\phi\,\vec{\nabla}\psi$$
$$\vec{A}\cdot\vec{n} = \phi\,\vec{\nabla}\psi\,\vec{n} \equiv \phi\,\partial_n\psi$$

$$\Rightarrow \int_V (\phi\,\vec{\nabla}^2\psi + \vec{\nabla}\phi\,\vec{\nabla}\psi)\,d^3x = \int_S \phi\,\vec{\nabla}\psi\cdot\vec{n}\,da$$

Green I

subtract same eq with ϕ and ψ interchanged

Green II: $$\int_V (\phi\,\vec{\nabla}^2\psi - \psi\,\vec{\nabla}^2\phi)\,d^3x' = \oint_S \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right)da'$$

choose ϕ a potential $\vec{\nabla}^2\phi = -4\pi\rho$

and ψ a Green's function $\psi = G(x', x)$

$$\nabla^2_{x'}\, G(\vec{x}', \vec{x}) = -4\pi\,\delta^3(x', x)$$

$$-4\pi\,\phi(x) + 4\pi\int \rho(x')\,G(x', x)\,d^3x'$$
$$= \oint_S \phi(x')\left(\vec{\nabla}_{x'}G(x', x) - G(x', x)\,\vec{\nabla}\phi\right)\cdot\vec{n}\,da$$

choose Green's function such that $G(x', x) = 0$ if $x' \in S$

$$\Rightarrow \phi(x) = \int \rho(x')\,G(x', x)\,d^3x' - \oint_S \phi(x')\,\vec{\nabla}_{x'}G(x', x)\cdot\vec{n}\,da$$

4. Energy and stress in an electrostatic Field

a) Energy

Work done to bring a small charge dq_i from ∞ to r_i

$$\delta W_i = (\phi(r_i) - \phi(\infty))\,\delta q_i$$
$$= -\int_{\infty}^{r_i} \vec{E}\cdot d\vec{s}\;\delta q_i$$

total work for many charges broaght from infinity

$$\delta W = \sum_i \delta W_i = \sum_i \phi(r_i)\,\delta q_i$$
$$= \int d^3r\;\phi(r)\,\delta\rho(r)$$

$$\vec{\nabla}\vec{E} = -4\pi\rho \qquad \Rightarrow \qquad \delta\rho = \frac{1}{4\pi}\vec{\nabla}\delta\vec{E}$$

$$\Rightarrow \delta W = \int d^3r\;\phi(r)\,\frac{1}{4\pi}\vec{\nabla}\delta E$$
$$= -\int d^3r\;\vec{\nabla}\phi\,\frac{1}{4\pi}\delta E$$

partial integration → we integrate over all space $\Rightarrow$ surface term vanishes

$$\Rightarrow \delta W = \frac{1}{4\pi}\int d^3r\;\vec{E}\cdot\delta E$$
$$= \frac{1}{8\pi}\int d^3r\;\delta\vec{E}^2$$

Work done to change the Field strength from $E_0 \rightarrow E_f$

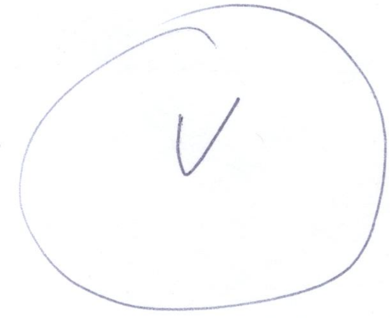
$$W = \frac{1}{8\pi}\int d^3r\,(E_f^2 - E_0^2)$$

$\vec{E}$ is changed by taking charges from ∞ to r_i

$\Rightarrow$ energy density is $\dfrac{\vec{E}^2}{8\pi}$

b) <u>Stress tensor</u>

Force on charges inside S



$$F_i = \int_V d^3r\, \rho(\vec{r})\, E_i(\vec{r})$$

$i = 1,2,3$ ← i'th component of $\vec{E}$ $\quad \partial V = S$

$$= \frac{1}{4\pi}\int_V d^3r\, \vec{\nabla}\cdot\vec{E}\, E_i$$

$$= \frac{1}{4\pi}\int_V d^3r \left(\partial_j (E_j E_i) - E_j \partial_j E_i\right)$$

$$\vec{\nabla}\times\vec{E} = 0 \Rightarrow \varepsilon_{ijk}(\partial_j E_k - \partial_k E_j) = 0$$

$$\Rightarrow \partial_j E_k - \partial_k E_j = 0 \quad k \neq j$$

and trivially $= 0$ for $k = j$

$$\Rightarrow E_j \partial_j E_i = E_j \partial_i E_j + \underbrace{E_j \partial_j E_i - E_j \partial_i E_j}_{=0}$$

$$= \tfrac{1}{2}\partial_i E_j^2$$

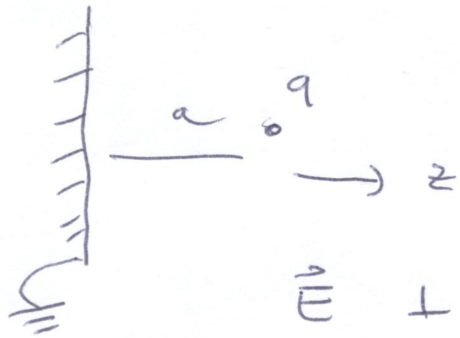
$$\Rightarrow F_i = \frac{1}{4\pi}\int_V d^3r\, \partial_j \left(E_j E_i - \delta_{ij}\frac{\vec{E}^2}{2}\right)$$

$$= \int_S dS_j\, T_{ji}$$

↑ Gauss

Maxwell stress tensor $\quad T_{ji} = \frac{1}{4\pi}\left(E_j E_i - \delta_{ij}\frac{\vec{E}^2}{2}\right)$

Example



$\vec{E} \perp$ surface at surface

$\Rightarrow \vec{E} = (0, 0, E_z)$

$$T_{ji}\Big|_s = \begin{pmatrix} -\frac{1}{2}E_z^2 & & \\ & -\frac{1}{2}E_z^2 & \\ & & \frac{1}{2}E_z^2 \end{pmatrix} \frac{1}{4\pi}$$

$$F_i = \int ds_i T_{ji}$$

$$= \int \vec{n}\, ds\, T_{33} = -\int_{ds} \frac{1}{8\pi} E_z^2$$

($\vec{n}$: normal to the outside)

E is the field due to the charge and image charge

$$\vec{E} = -\frac{q}{(a^2+\rho^2)}\, 2\cos\varphi$$

$$= -\frac{2qa}{(a^2+\rho^2)^{3/2}}$$

$$\vec{F} = -\int ds \frac{1}{8\pi} E_z^2 = -\int ds \frac{1}{8\pi} \frac{4q^2a^2}{(a^2+\rho^2)^3}$$

$\int \rho\, d\rho\, d\varphi$

$$= -\frac{2\pi\, 4}{8\pi} \int_0^\infty \rho\, d\rho \frac{q^2a^2}{(a^2+\rho^2)^3} = -\frac{q^2}{a^2} \int_0^\infty \frac{\rho\, d\rho}{(1+\rho^2)^3}$$

$$= -\frac{q^2}{4a^2}$$ correct

Energy in terms of charges

$$W_E = \frac{1}{8\pi} \int_V \vec{E}^2 d^3r$$

$$= \frac{1}{8\pi} \int_V \vec{\nabla}\phi \, \vec{\nabla}\phi \, d^3r$$

$$= -\frac{1}{8\pi} \int_V \phi \, \vec{\nabla}^2 \phi \, d^3r$$

$$= \frac{1}{2} \int d^3r \, \phi(r) \rho(r)$$

$$= \frac{1}{2} \int d^3r d^3r' \frac{\rho(r)\rho(r')}{|\vec{r} - \vec{r}'|}$$

↑ factor $\frac{1}{2}$ because this is the work energy to bring charges from infinity.

For a collection of point charge

$$W = \frac{1}{2} \sum_{i \neq j} \frac{q_i q_j}{|\vec{r}_i - \vec{r}_j|}$$

the energy is always because of the field from another charge.

we see later that the energy of a magnetic field is $W_B = \frac{1}{8\pi} \int d^3r \, \vec{B}^2$

$$T + W_E + W_B = \text{constant}$$

↑ Kinetic energy

5) Electrostatics in matter

- in conductors electrons can move freely
- in dielectric the electrons are bound to the atoms

in electric field the charge distribution of the electrons gets displaced wrt the nucleus and the atom gets a dipole moment.

model This effect can be approximated by a dipole density $\vec{p}(r)$

This gives the potential

$$\phi(r) = \sum_i \frac{\vec{p}_i (\vec{r} - \vec{r}_i)}{|(\vec{r} - r_i|^3}$$

$$= -\sum p_i \vec{\nabla}_{\vec{r}} \frac{1}{|\vec{r} - \vec{r}_i|}$$

$$= -\int d^3r' \, \vec{p}(r') \vec{\nabla}_{\vec{r}} \frac{1}{|\vec{r} - \vec{r}'|}$$

$$\vec{\nabla}_{\vec{r}} = -\vec{\nabla}_{r'}$$

$$= \int d^3r' \left(\nabla_{r'} \left(p(r') \frac{1}{|\vec{r} - \vec{r}'|} \right) - \frac{(\nabla_{r'} p(r'))}{|r - r'|} \right)$$

$$= \int_{S=\partial V} \frac{\vec{P}(r') \cdot d\vec{a}}{|r - r'|} + \int d^3 r' \frac{(-\vec{\nabla}_{\vec{r}'} P(\vec{r}'))}{|r - r'|}$$

↑ potential of surface charge density on S

$\sigma = \vec{P} \cdot \vec{n}$

induced charge density

$\rho_b = -\vec{\nabla}_{r'} P(r')$

- if the dipole density is constant then $\vec{\nabla}_{r'} \vec{P}(r') = 0$ and we only have a surface charge density

effect of induced charge density

$$\vec{\nabla} \vec{E} = 4\pi(\rho + \rho_{ind})$$

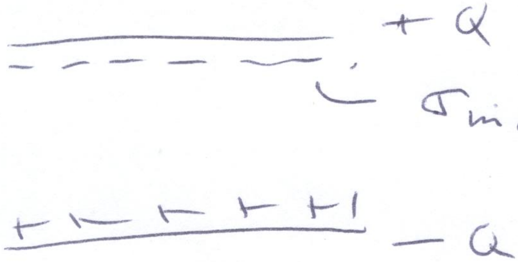
$$= 4\pi(\rho - \vec{\nabla} \vec{P})$$

$$\Rightarrow \vec{\nabla}(\underbrace{\vec{E} + 4\pi\vec{P}}_{\equiv \vec{D}}) = 4\pi \rho$$

(ρ ↑ free charges)

For linear media we have $\vec{D} = \varepsilon \vec{E}$

For a surface charge density we have:



$\sigma_{ind} = (\vec{P} \cdot n)$

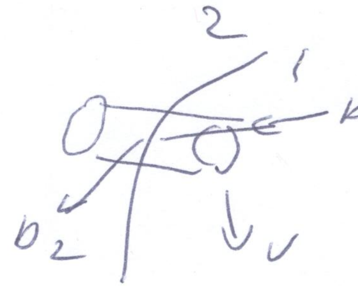
$$\vec{E} = 4\pi(\sigma - \sigma_{ind}) \cdot \vec{n}$$

$$= 4\pi(\sigma - (\vec{P} \cdot \vec{n}))\vec{n}$$

$$\Rightarrow (\vec{E} + 4\pi\vec{P}) = 4\pi\sigma\vec{n}$$

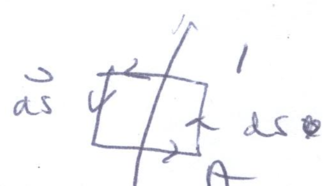
5b) <u>Boundary conditions in a medium</u>

$$\int_{\partial V} \vec{D} \cdot \vec{n}\, da = 4\pi\sigma A$$



$$\Rightarrow D_{2n} - D_{1n} = 4\pi\sigma$$

$$\nabla \vec{E} = 0$$



$$\int_{\partial A} \vec{E} \cdot d\vec{s} = \int \vec{\nabla} \times \vec{E} \cdot d\vec{A} = 0$$

$$E^1_{tang} - E^2_{tang} = 0$$

Electrostatic energy in a medium

We do this again by building up a charge density

Work done

$$\delta W = \sum_i \delta q_i \, \phi(\vec{r}_i)$$

$$= \int d^3r \, \delta\rho(\vec{r}) \, \phi(\vec{r})$$

$$= \int d^3r \, \frac{\vec{\nabla}\cdot\delta\vec{D}}{4\pi} \, \phi(r)$$

$$= -\int d^3r \, \frac{\delta\vec{D}}{4\pi} \cdot \vec{\nabla}\phi$$

$$= \int d^3r \, \frac{\delta\vec{D}\cdot\vec{E}}{4\pi}$$

for a linear medium $\qquad \vec{D}_i = \varepsilon_{ij} E_j$

$$= \frac{1}{4\pi}\int d^3r \, \varepsilon_{ij} \, \delta E_j \, E_i$$

$$\varepsilon_{ij} = \varepsilon^S_{ij} + \varepsilon^A_{ij} \qquad \varepsilon^S_{ij} = \varepsilon_{ji}$$

$$\varepsilon^A_{ij} = -\varepsilon_{ji}$$

$$\frac{1}{4\pi}\int d^3r \, \varepsilon^S_{ij} \, \delta E_j E_i = \frac{1}{4\pi}\int d^3r \, \varepsilon^S_{ij} \, \frac{1}{2}\delta(E_i E_j)$$

$$= \frac{1}{8\pi}\int d^3r \, \delta(\vec{D}^S\cdot\vec{E})$$

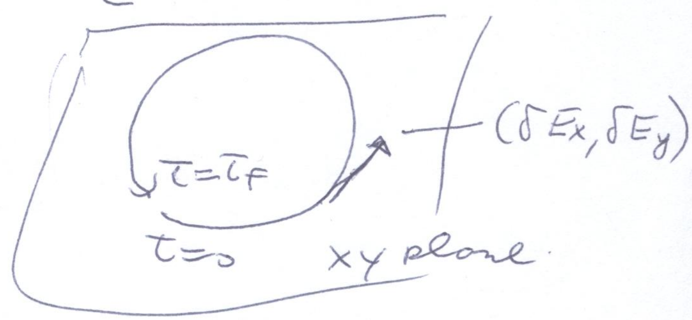
$$\Rightarrow W = \frac{1}{8\pi}\int d^3r \, \vec{D}^S\cdot\vec{E}$$

For the anti-symmetric part of the dielectric constant we obtain

$$\delta W = \frac{1}{4\pi}\int d^3r\, \varepsilon^A_{ij}\, \frac{1}{2}\left(\delta E_j \cdot E_i - \delta E_i \cdot E_j\right)$$

We now let the electric Field depend on a parameter τ

$$\vec{E} \to \vec{E}(\tau)$$



and calculate the work after $\vec{E}$ returns again to the same value

$$\oint \delta W\, d\tau = \frac{1}{4\pi}\int d^3r\, \frac{\varepsilon^A_{ij}}{2}\oint d\tau\left(\delta E_j E_i - \delta E_i E_j\right)$$

We consider $\vec{E}$ in the xy plane.

$$= \frac{1}{4\pi}\int d^3r\, \frac{\varepsilon^A_{xy}}{2}\oint d\tau\left(\delta E_y E_x - \delta E_x E_y\right)$$

$$\delta E_y = \frac{dE_y}{d\tau} \qquad \delta E_x = \frac{dE_x}{d\tau}$$

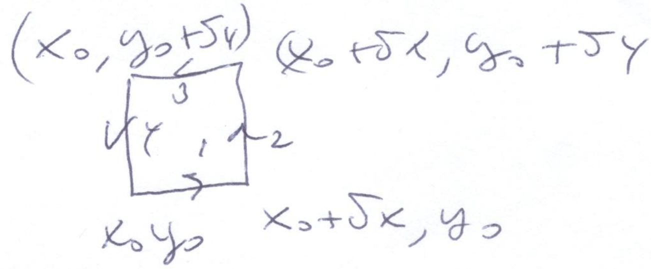
$$\oint\left(E_x\, dE_y - E_y\, dE_x\right)$$

This is in E_x, E_y space

Let us look how this work for

x, y : $\frac{1}{2}\oint(x\, dy - y\, dx$

We consider an infinitesimal loop



$$\frac{1}{2}\int (x\,dy - y\,dy)$$

$$\frac{1}{2}\Big(-y_0\,\delta x + (x_0+\delta x)\,\delta y$$

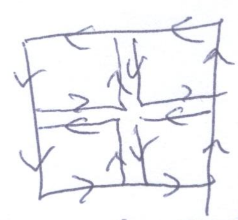
① ②

$$(-)(-)(y_0+\delta y)\,\delta x \quad - x_0\,\delta y$$

③ ④

$$= \frac{1}{2}\;2\,\delta x\,\delta y \quad = \text{area of loop}$$

For bigger loop



integral over the inner loops cancels.

$\Rightarrow$ $\delta W \neq 0$ when $\vec{E}$ returns to its original value.

$\Rightarrow$ Σ^{A}_{ij} describes absorbtion