Integral form of Gauss law

$$\int_V dv\, \vec{\nabla}\cdot\vec{E} = \int dV\, 4\pi\rho$$

$$\parallel$$

$$\int_{\partial V} da\, \vec{n}\cdot\vec{E}$$

e) Poisson law

$$\left.\begin{aligned}\vec{\nabla}\vec{E} &= 4\pi\rho \\ \vec{E} &= -\vec{\nabla}\phi\end{aligned}\right\} \Rightarrow \vec{\nabla}^2\phi = -4\pi\rho$$

Poisson equatia

$\rho = 0 \quad \vec{\nabla}^2\phi = 0$ Laplace eq

solutions are called harmonic function.

f) Uniqueness

A vector field that satisfies

$\vec{\nabla}\vec{E} = 0 \qquad \vec{\nabla}\times\vec{E}$

and vanishes at least as $\frac{1}{r^2}$ for $r\to\infty$

vanishes everywhere

This follows from liouville's theoreme:

if f is a harmonic function on $\mathbb{R}^n$

and f is bounded from above or below

then f is constant

Reason $\quad \vec{E} = -\vec{\nabla}\phi$

$$\vec{\nabla}^2\phi = 0$$

$\vec{E} \sim \frac{1}{r^2}$ for $r \to \infty \Rightarrow \phi \sim \frac{1}{r}$ for $r \to \infty$

$\vec{\nabla}.\vec{E} = 0 \Rightarrow \exists$ no singularities

$\Rightarrow \phi$ is bounded $\Rightarrow \phi =$ constant

$\Rightarrow \vec{E} = 0$

Consequence: uniqueness theorem

$$\left.\begin{array}{l} \vec{\nabla}.\vec{E}_1 = 4\pi\rho_1 \\ \vec{\nabla}.\vec{E}_2 = 4\pi\rho_2 \end{array}\right\} \Rightarrow \vec{\nabla}(\vec{E}_1 - E_2) = 0$$

statics $\Rightarrow \quad \vec{\nabla}\times(\vec{E}_1 - \vec{E}_2) = 0$

$$\Rightarrow \quad E_1 - E_2 = 0$$

Solutions of the Laplace eq

$$\vec{\nabla}^2\phi = 0 \qquad \begin{array}{l} z = r\cos\theta \\ x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \end{array}$$

spherical coordinates

$$\vec{\nabla}^2\phi = \frac{1}{r}\partial_r^2(r\phi) + \frac{1}{r^2\sin\theta}\partial_\theta\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2}$$

$$\phi = \frac{u(r)}{r}P(\theta)Q(\varphi)$$

separation of variables

$$\nabla^2\phi = \frac{PQ}{r}\partial_r^2 u + \frac{uQ}{rr^2\sin\theta}\partial_\theta(\sin\theta\,\partial_\theta P)\,\underline{g}$$

$$+ \frac{u\;P}{rr^2\sin^2\theta}\frac{\partial^2 Q}{\partial\varphi^2} = 0$$

divide by uPQ

$$\frac{u^{-1}}{r}\partial_r^2 u + \frac{P^{-1}}{rr^2\sin\theta}\partial_\theta(\sin\theta\,\partial_\theta P)$$

$$+ \frac{Q^{-1}}{rr^2\sin^2\theta}\frac{\partial^2 Q}{\partial\varphi^2} = 0$$

$\times r^3$

$$r^2 u^{-1}\partial_r^2 u + \frac{P^{-1}}{\sin^2\theta}\partial_\theta(\sin\theta\,\partial_\theta P)$$

$$+ \frac{Q^{-1}}{\sin^2\theta}\frac{\partial^2 Q}{\partial\varphi^2} = 0$$

only depends on r

only depends on θ, φ

$\Rightarrow$ both should be constant because the equation is valid for all r, θ, φ

$$r^2 u^{-1}\partial_r^2 u = c \Rightarrow \partial_r^2 u = c\frac{u}{r^2}$$

$$u = \alpha r^{l+1} \Rightarrow l(l+1)r^{l-2} = c\,r^{l-2}$$

$$\Rightarrow c = l(l+1)$$

then

$$\ell(\ell+1) \sin^2\theta + P^{-1}\partial_\theta(\sin\theta\,\partial_\theta P)$$

$$+ Q^{-1}\frac{\partial Q^2}{\partial \varphi^2} = 0$$

$$\Rightarrow \ell(\ell+1)\sin^2\theta + P^{-1}\partial_\theta(\sin\theta\,\partial_\theta P)$$

$$= -\left(\frac{\partial Q^2}{\partial \varphi^2}\right) Q^{-1}$$

function of θ function of φ

equal $\forall\, \theta, \varphi \Rightarrow$ they have to be constant

$$\Rightarrow Q^{-1}\frac{\partial^2 Q^2}{\partial \varphi^2} = -m^2 \Rightarrow Q = e^{\pm i m \varphi}$$

$$\Rightarrow -\partial_\theta(\sin\theta\,\partial_\theta P) = \ell(\ell+1)\sin^2\theta\, P - m^2 P$$

This equation is known as the Legendre equation.

The solutions are given by the associated Legendre polynomials

The combination PQ is given by $PQ = c' P_{\ell m}(\cos\theta)\, e^{im\varphi} \equiv \frac{1}{c}\, Y_{\ell m}(\theta, \varphi)$

spherical harmonics

The constant c' is fixed by the normalization sum rule

$$\sum_{m=-\ell}^{\ell} Y_{\ell m}(\theta, \varphi)\, Y^{*}_{\ell m}(\theta, \varphi) = \frac{2\ell+1}{4\pi}$$

$$Y_{00} = \frac{1}{\sqrt{4\pi}}$$

$$Y_{11} = -\sqrt{\frac{3}{8\pi}}\, \sin\theta\, e^{i\phi} = -\sqrt{\frac{3}{8\pi}}\, \frac{x+iy}{r}$$

$$Y_{10} = \sqrt{\frac{3}{4\pi}}\, \cos\theta = \sqrt{\frac{3}{4\pi}}\, \frac{z}{r}$$

$$Y_{1-1} = \sqrt{\frac{3}{8\pi}}\, \sin\theta\, e^{-i\varphi} = \sqrt{\frac{3}{8\pi}}\, \frac{x-iy}{r}$$

We have found an infinite set of harmonic functions

$$\nabla^2\, r^{\ell}\, Y_{\ell m}(\theta, \varphi) = 0$$

$$\ell(\ell+1) = (-\ell-1)((-\ell-1)+1)$$

$\Rightarrow u(r) = \frac{1}{r^{\ell}}$ is also solution

$$\Rightarrow \quad \vec{\nabla}^2 \frac{1}{r^{\ell+1}} Y_{\ell m}(\theta, \varphi) = 0$$

Most general solution of the Laplace eq.

$$\phi(r,\theta,\varphi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left(A_{\ell m} r^{\ell} + B_{\ell m} r^{-\ell-1} \right) Y_{\ell m}(\theta,\varphi)$$

This is a complete set of functions

Strategy for solution of Poisson eq

$$\nabla^2 \phi = -4\pi\rho$$

then $\nabla^2(\phi + \phi_h) = -4\pi\rho$

special solution

$$\vec{\nabla}^2 \phi_h = 0$$

Generally we have a charge distribution and boundary conditions

The ϕ_h are determined by the boundary conditions.

outside charge distribution ; $\rho = 0$

$\Rightarrow \vec{\nabla}^2 \phi = 0$ $\quad$ $\phi \to 0$ for $r \to \infty \Rightarrow A_{\ell m} = 0$

inside charge distribution : solution should be regular $\Rightarrow B_{\ell m} = 0$

Some properties of spherical harmonics

$$Y_{lm}(\theta,\varphi) = \sqrt{\frac{2l+1}{4\pi}}\sqrt{\frac{(l-m)!}{(l+m)!}}\, P_{lm}(\cos\theta)\, e^{im\varphi}$$

$$r^2 \vec{\nabla}^2 Y_{lm}(\theta,\varphi) = -l(l+1) Y_{lm}(\theta,\varphi)$$

$$Y_{l-m}(\theta,\varphi) = (-1)^m Y^*_{lm}(\theta,\varphi)$$

orthogonality

$$\int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta\, d\theta\, Y^*_{l'm'}(\theta,\varphi) Y_{lm}(\theta,\varphi) = \delta_{ll'}\delta_{mm'}$$

completeness

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y^*_{lm}(\theta',\varphi') Y_{lm}(\theta,\varphi) = \delta(\varphi-\varphi') \times \delta(\cos\theta - \cos\theta')$$

$\Rightarrow$ An arbitrary function of θ and φ can be expressed as

$$f(\theta,\varphi) = \sum a_{lm} Y_{lm}(\theta,\varphi)$$

sum rule

$$\sum_{m=-l} Y_{lm}(\theta,\varphi) Y^*_{lm}(\theta,\varphi) = \frac{2l+1}{4\pi}$$

Example 1: single charge at 0

$$\rho(r) = q\,\delta^3(r)$$

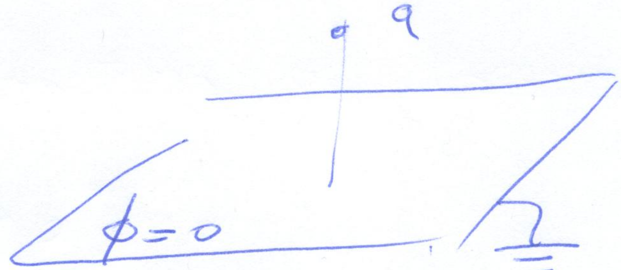
potential is spherically symmetric

$\Rightarrow$ only $\ell = 0$, $m = 0$ is allowed

$$\Rightarrow \quad \phi = A_{00} + B_{00}\, r^{-1}$$

$\phi \to 0$ for $r \to \infty$ $\Rightarrow A_{00} = 0$

Example 2



solution is axial symmetric

$\Rightarrow m = 0$ $\qquad$ $\phi \to 0$ for $r \to \infty$

$\Rightarrow A_{\ell m} = 0$ for $r > 0$

$$\Rightarrow \quad \phi = \sum B_{\ell m}\, r^{-\ell-1}\, Y_{\ell 0}(\theta, \varphi)$$

does not depend on φ.

$z = 0 \quad \theta = \frac{\pi}{2}$

$$\sum B_{\ell m}\, r^{-\ell-1}\, Y_{\ell 0}\left(\tfrac{\pi}{2}, \varphi\right) = 0$$

only odd ℓ are possible

$$Y_{2k+1,0}\left(\tfrac{\pi}{2}, \varphi\right) = 0$$

$$\Rightarrow \phi = \sum_{\ell \text{ odd}} a_\ell\, r^{-\ell-1}\, \underbrace{Y_{(\ell,0)}(\theta,\varphi)}_{= \sqrt{\frac{2\ell+1}{4\pi}}}$$

look at the z-axis $\theta_\phi = 0$

$$\Rightarrow \phi = \sum_{\ell \text{ od}} b_\ell\, r^{-\ell-1}$$

$$= \sum_\ell \frac{\left(-(-1)^\ell + 1\right)}{2} b_\ell\, r^{-\ell-1}$$

for

should behaves as $\frac{q}{|z-a|}$ for $z \to a$

$a_\ell = \sqrt{\frac{4\pi}{2\ell+1}}\, b_\ell$

$$\Rightarrow b_\ell = q a^\ell\, 2 \quad \text{geometric series}$$

on z axis $\phi = \frac{q}{(r-a)} - \frac{q}{r+a}$

$$\Rightarrow \phi = \sum \left(1 - (-1)^\ell\right) \frac{q a^\ell}{r^{\ell+1}}\, Y_{\ell 0}(\theta,\varphi) \frac{\sqrt{\pi 4}}{\sqrt{2\ell+1}}$$

potential on z-axis

$$\phi = \sum_{\ell=0}^{\infty} \left((1) - (-1)^{\ell}\right) \frac{q a^{\ell}}{r^{\ell+1}} \underbrace{Y_{\ell 0}(0,\varphi) \sqrt{\frac{4\pi}{2\ell+1}}}_{=1}$$

$$= \frac{q}{r} \frac{1}{1-\frac{a}{r}} - \frac{q}{r} \frac{1}{1+\frac{a}{r}}$$

$$= \frac{q}{r-a} - \frac{q}{r+a}$$

$\uparrow$

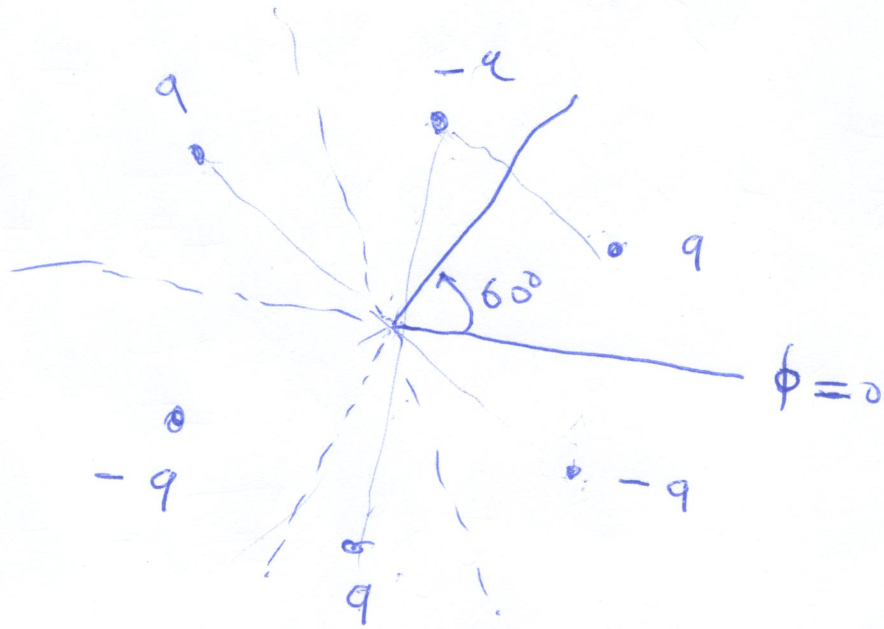
image charge

This problem can be solved by image charges.
The trick is to Find other charges that give the same boundary condition

$\Rightarrow$ Solution on the right of the plane is the same, not on the left

$$\phi(\vec{r}) = \frac{q}{|\vec{r}-\vec{a}|} - \frac{q}{|\vec{r}+\vec{a}|}$$

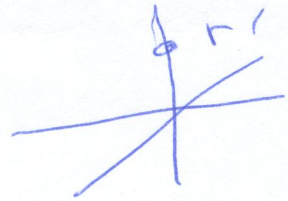
2nd Example of image charges



Expansion of $\frac{1}{|\vec{r}-\vec{r}\,'|}$ in spherical harmonics

choose $|\vec{r}| > |r'|$

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \sum \frac{1}{r^{\ell+1}} A_{\ell m} P_\ell Y_{\ell m}(\theta, \varphi)$$



choose $\vec{r}\,'$ on z axis

on z axis: $\frac{1}{|r-r'|} = \sum_{k=0}^{\infty} \frac{r'^k}{r^{k+1}}$

~~th~~ no dependence on $\varphi \Rightarrow m = 0$

$$\Rightarrow A_{\ell 0} = \frac{r'^\ell}{Y_{\ell 0}(0, \varphi)} = r'^\ell \left(\frac{4\pi}{2\ell+1}\right)^{1/2} Y_{\ell 0}(0, \varphi)$$

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \sum \frac{1}{r^{\ell+1}} r'^\ell \frac{4\pi}{2\ell+1} Y_{\ell 0}(0, \varphi) Y_{\ell 0}(\theta, \varphi)$$

$Y_{\ell 0}(\theta', \varphi) \, Y_{\ell 0}(\theta, \varphi)$

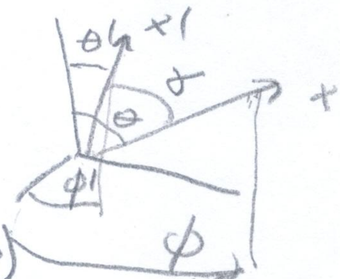
only depends on angle between $\vec{r}$ and $\vec{r}\,'$

$$Y_{\ell 0}(\theta, \varphi) = \sqrt{\frac{2\ell+1}{4\pi}} P_\ell(\cos\theta)$$

$$Y_{\ell 0}(0, 0) = \sqrt{\frac{2\ell+1}{4\pi}}$$

addition theorem

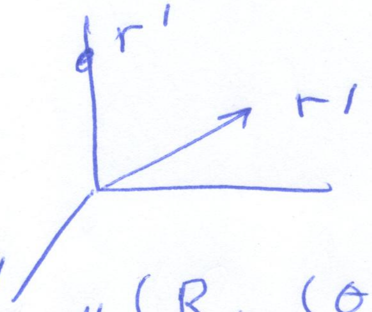
$$P_\ell(\cos\gamma) = \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{\ell} Y^*_{\ell m}(\theta', \varphi') \times Y_{\ell m}(\theta, \varphi)$$



$$\Rightarrow \frac{1}{|\vec{r}-\vec{r}\,'|} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{4\pi}{(2\ell+1)} \frac{r'^\ell}{r^{\ell+1}} Y^*_{\ell m}(\theta', \varphi') Y_{\ell m}(\theta, \varphi)$$

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \sum_{\ell} \frac{r'^{\ell}}{r^{\ell+1}} \sqrt{\frac{4\pi}{2\ell+1}}\, Y_{\ell 0}(\theta, \varphi)$$

now rotate $r' \to (\theta', \varphi')$



then $$\underset{\downarrow\, 0}{Y_{\ell m}}(\vec{r}) = \sum_{m''} \mathcal{D}^{\ell}_{m''\underset{\downarrow\, 0}{m}}\, Y_{\ell m''}\big(\underbrace{R_{\theta'\varphi'}(\theta, \varphi)}_{\text{new } \theta, \varphi \text{ coordinates of } \vec{r}}\big)$$

Wigner $\mathcal{D}$ matrices

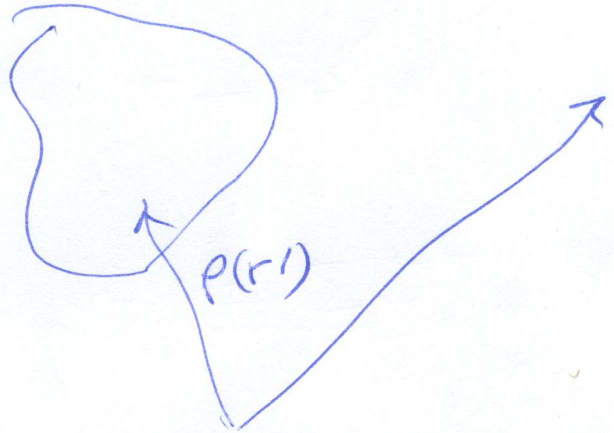
$$\mathcal{D}^{\ell}_{m''0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y^{*}_{\ell m''}(\theta', \varphi')$$

$$\Rightarrow \frac{1}{|\vec{r}-\vec{r}\,'|} = \sum_{\ell, m''} \frac{r'^{\ell}}{r^{\ell+1}} \frac{4\pi}{2\ell+1}\, Y^{*}_{\ell m''}(\theta', \varphi')\, Y_{\ell m''}(\theta, \varphi)$$

note that $r > r'$

Multipole expansion

a)



$|\vec{r}| > |r'|$

$$\phi(r'') = \int d^3r'\, \rho(r') \frac{1}{|\vec{r}\,' - \vec{r}|}$$

$$= \int d^3r'\, \rho(r') \sum_{lm} \frac{r'^l}{r^{l+1}} \frac{4\pi}{2l+1} \rho(r') Y^*_{lm}(\theta', \varphi') \times Y_{lm}(\theta, \varphi)$$

multipole moment

$$Q_{lm} = \int d^3r'\, r'^l \rho(\vec{r}\,') Y^*_{lm}(\theta' \varphi')$$

$$\Rightarrow \phi(r) = \sum_{l=0}^{\infty} \frac{4\pi}{2l+1} \frac{1}{r^{l+1}} \sum_{m=-l}^{l} Y_{lm}(\theta, \varphi) Q_{lm}$$

$l=0$ monopole
$l=1$ dipole
$l=2$ quadrapole

b) Dipole moment

$$\vec{p} = \int d^3x' \, \rho(x') \, \vec{x}'$$

$$Q_{10} = \sqrt{\frac{3}{4\pi}} \int z' \rho(x') \, d^3x' = \sqrt{\frac{3}{4\pi}} \, p_z$$

$$Q_{11} = -\sqrt{\frac{3}{8\pi}} \int (x' - iy') \, \rho(x') \, d^3x'$$

$$= -\sqrt{\frac{3}{8\pi}} (p_x - i p_y)$$

$$Q_{1-1} = - Q_{11}^* = \sqrt{\frac{3}{8\pi}} (p_x + i p_y)$$

The potential of a dipole is given by

$$\phi(r) = \frac{4\pi}{\underset{2\ell+1}{3}} \frac{1}{r^2} \Big(Y_{11} \left(-\sqrt{\frac{3}{8\pi}}\right) (p_x - i p_y)$$

$$+ Y_{10} \sqrt{\frac{3}{4\pi}} \, p_z$$

$$+ Y_{1-1} \sqrt{\frac{3}{8\pi}} (p_x + i p_y) \Big)$$

$$Y_{11} = -\sqrt{\frac{3}{8\pi}} (x + iy)/r \qquad Y_{10} = \sqrt{\frac{3}{4\pi}} \frac{z}{r}$$

$$Y_{1-1} = \sqrt{\frac{3}{8\pi}} \frac{(x - iy)}{r}$$

$$\phi(r) = \frac{4\pi}{3} \frac{1}{r^3} \Big((x+iy)(p_x - i p_y) \frac{3}{8\pi} + z \, p_z \frac{3}{4\pi}$$

$$+ (x - iy)(p_x + i p_y) \frac{3}{8\pi}$$

$$= \frac{1}{r^3} (x p_x + y p_y + z p_z) = \frac{1}{r^3} (\vec{r} \cdot \vec{p})$$