Integral form of Gauss law

$$\int_V dV\, \vec{\nabla}\cdot\vec{E} = \int dV\, 4\pi\rho$$

$$\parallel$$

$$\int_{\partial V} da\, \vec{n}\cdot\vec{E}$$

e) Poisson law

$$\left.\begin{aligned} \vec{\nabla}\vec{E} &= 4\pi\rho \\ \vec{E} &= -\vec{\nabla}\phi \end{aligned}\right\} \Rightarrow \vec{\nabla}^2\phi = -4\pi\rho$$

Poisson equatia

$\rho = 0 \quad \vec{\nabla}^2\phi = 0$ Laplace eq

solutions are called harmonic function.

f) Uniqueness

A vector field that satisfies

$\vec{\nabla}\vec{E} = 0 \quad \vec{\nabla}\times\vec{E}$

and vanishes at least as $\frac{1}{r^2}$ for $r\to\infty$

vanishes everywhere

This follows from liouville's theoreme:

if f is a harmonic function on $\mathbb{R}^n$

and f is bounded from above or below

then f is constant

Reason $\vec{E} = -\vec{\nabla}\phi$

$$\vec{\nabla}^2 \phi = 0$$

$\vec{E} \sim \frac{1}{r^2}$ for $r \to \infty \Rightarrow \phi \sim \frac{1}{r}$ for $r \to \infty$

$\vec{\nabla} \cdot \vec{E} = 0 \Rightarrow \exists$ no singularities

$\Rightarrow \phi$ is bounded $\Rightarrow \phi =$ constant

$\Rightarrow \vec{E} = 0$

Consequence: uniqueness theorem

$$\left.\begin{aligned} \vec{\nabla} \cdot \vec{E}_1 &= 4\pi \rho_1 \\ \vec{\nabla} \cdot \vec{E}_2 &= 4\pi \rho_2 \end{aligned}\right\} \Rightarrow \vec{\nabla}(\vec{E}_1 - E_2) = 0$$

statics $\Rightarrow \vec{\nabla} \times (\vec{E}_1 - \vec{E}_2) = 0$

$\Rightarrow E_1 - E_2 = 0$

Solutions of the Laplace eq

$\vec{\nabla}^2 \phi = 0$

$$z = r\cos\theta$$
$$x = r\sin\theta\cos\varphi$$
$$y = r\sin\theta\sin\varphi$$

spherical coordinates

$$\vec{\nabla}^2 \phi = \frac{1}{r}\partial_r^2 (r\phi) + \frac{1}{r^2 \sin\theta}\partial_\theta\left(\sin\theta \frac{\partial \phi}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \phi}{\partial \varphi^2}$$

$$\phi = \frac{u(r)}{r} P(\theta) Q(\varphi)$$

sepration of variables

$$\nabla^2\phi = \frac{PQ}{r}\partial_r^2 u + \frac{uQ}{rr^2\sin\theta}\partial_\theta(\sin\theta\,\partial_\theta P)\,\underline{g}$$

$$+ \frac{uP}{rr^2\sin^2\theta}\frac{\partial^2 Q}{\partial\varphi^2} = 0$$

divide by uPQ

$$\frac{u^{-1}}{r}\partial_r^2 u + \frac{P^{-1}}{rr^2\sin\theta}\partial_\theta(\sin\theta\,\partial_\theta P)$$

$$+ \frac{Q^{-1}}{rr^2\sin^2\theta}\frac{\partial^2 Q}{\partial\varphi^2} = 0$$

$\times r^3$

$$r^2 u^{-1}\partial_r^2 u + \frac{P^{-1}}{\sin^2\theta}\partial_\theta(\sin\theta\,\partial_\theta P)$$

$$+ \frac{Q^{-1}}{\sin^2\theta}\frac{\partial^2 Q}{\partial\varphi^2} = 0$$

only depends on r

only depends on θ, φ

$\Rightarrow$ both should be constant because the equation is valid for all r, θ, φ

$$r^2 u^{-1}\partial_r^2 u = c \Rightarrow \partial_r^2 u = c\frac{u}{r^2}$$

$$u = \alpha r^{\ell+1} \Rightarrow \ell(\ell+1) r^{\ell-2} = c\, r^{\ell-2}$$

$$\Rightarrow c = \ell(\ell+1)$$

then

$$\ell(\ell+1)\sin^2\theta + P^{-1}\partial_\theta(\sin\theta\,\partial_\theta P)$$

$$+Q^{-1}\frac{\partial Q^2}{\partial\varphi^2} = 0$$

$$\Rightarrow \ell(\ell+1)\sin^2\theta + P^{-1}\partial_\theta(\sin\theta\,\partial_\theta P)$$

$$= -\left(\frac{\partial Q^2}{\partial\varphi^2}\right)Q^{-1}$$

function of θ function of φ

equal $\forall\,\theta,\varphi \Rightarrow$ they have to be constant

$$\Rightarrow Q^{-1}\frac{\partial^2 Q^2}{\partial\varphi^2} = -m^2 \Rightarrow Q = e^{\pm im\varphi}$$

$$\Rightarrow -\partial_\theta(\sin\theta\,\partial_\theta P) = \ell(\ell+1)\sin^2\theta\,P - m^2 P$$

This equation is known as the Legendre equation.

The solutions are given by the associated Legendre polynomials

The combination PQ is given by $PQ = c' P_{\ell m}(\cos\theta)\, e^{im\varphi} \equiv c' Y_{\ell m}(\theta, \varphi)$

spherical harmonics

The constant c' is fixed by the normalization sum rule

$$\sum_{m=-\ell}^{\ell} Y_{\ell m}(\theta,\varphi)\, Y^{*}_{\ell m}(\theta,\varphi) = \frac{2\ell+1}{4\pi}$$

$$Y_{00} = \frac{1}{\sqrt{4\pi}}$$

$$Y_{11} = -\sqrt{\frac{3}{8\pi}} \sin\theta\, e^{i\phi} = -\sqrt{\frac{3}{8\pi}} \frac{x+iy}{r}$$

$$Y_{10} = \sqrt{\frac{3}{4\pi}} \cos\theta = \sqrt{\frac{3}{4\pi}} \frac{z}{r}$$

$$Y_{1-1} = \sqrt{\frac{3}{8\pi}} \sin\theta\, e^{-i\varphi} = \sqrt{\frac{3}{8\pi}} \frac{x-iy}{r}$$

We have found an infinite set of harmonic functions

$$\nabla^2 r^{\ell}\, Y_{\ell m}(\theta,\varphi) = 0$$

$$\ell(\ell+1) = (-\ell-1)((-\ell-1)+1)$$

$\Rightarrow u(r) = \frac{1}{r^{\ell}}$ is also solution

$$\Rightarrow \quad \vec{\nabla}^2 \frac{1}{r^{\ell+1}} Y_{\ell m}(\theta, \varphi) = 0$$

Most general solution of the Laplace eq.

$$\phi(r,\theta,\varphi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left(A_{\ell m} r^{\ell} + B_{\ell m} r^{-\ell-1} \right) Y_{\ell m}(\theta, \varphi)$$

This is a complete set of functions

Strategy for solution of Poisson eq

$$\nabla^2 \phi = -4\pi\rho$$

then $\nabla^2(\phi + \phi_h) = -4\pi\rho$

special solution

$$\vec{\nabla}^2 \phi_h = 0$$

Generally we have a charge distribution and boundary conditions. The ϕ_h are determined by the boundary conditions.

outside charge distribution; $\rho = 0$

$\Rightarrow \vec{\nabla}^2 \phi = 0$ $\quad$ $\phi \to 0$ for $r \to \infty \Rightarrow A_{\ell m} = 0$

inside charge distribution: solution should be regular $\Rightarrow B_{\ell m} = 0$

<u>Example 1</u>: single charge at 0

$$\rho(r) = q\,\delta^3(r)$$

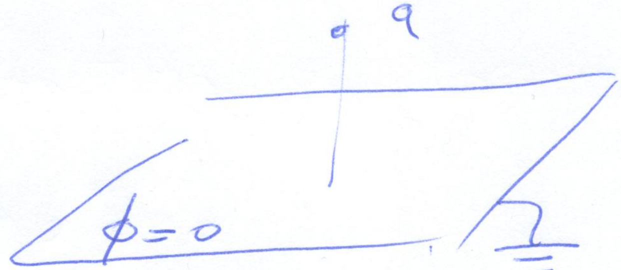
potential is spherically symmetric

$\Rightarrow$ only $\ell = 0$, $m = 0$ is allowed

$$\Rightarrow \quad \phi = A_{00} + B_{00}\, r^{-1}$$

$$\phi \to 0 \text{ for } r \to \infty \Rightarrow A_{00} = 0$$

<u>Example 2</u>



solution is axial symmetric

$\Rightarrow m = 0$ $\qquad$ $\phi \to 0$ for $r \to \infty$

$\Rightarrow A_{\ell m} = 0$ for $r > 0$

$$\Rightarrow \quad \phi = \sum B_{\ell m}\, r^{-\ell-1}\, Y_{\ell 0}(\theta, \varphi)$$

does not depend on φ.

$z = 0 \quad \theta = \frac{\pi}{2}$

$$\sum B_{\ell m}\, r^{-\ell-1}\, Y_{\ell 0}\left(\frac{\pi}{2}, \varphi\right) = 0$$

only odd ℓ are possible

$$Y_{2k+1,0}\left(\frac{\pi}{2}, \varphi\right) = 0$$

$$\Rightarrow \phi = \sum_{\ell\ \text{odd}} a_\ell\, r^{-\ell-1}\, Y_{(\ell,0)}(\theta,\varphi)$$

look at the z-axis $\theta_0 \Rightarrow$

$$\Rightarrow \phi = \sum_{\ell\ \text{odd}} b_\ell\, r^{-\ell-1}$$

for $$= \sum_\ell \frac{\left(-(-1)^\ell+1\right)}{2}\, b_\ell\, r^{-\ell-1}$$

should behaves as $\dfrac{q}{|z-a|}$ for $z \to a$

$$\Rightarrow b_\ell = q a^\ell\, 2 \quad \text{geometric series}$$

on z axis $\phi = \dfrac{q}{|r-a|} - \dfrac{q}{r+a}$

$$\Rightarrow \phi = \sum \left(1-(-1)^\ell\right) \frac{q a^\ell}{r^{\ell+1}}\, Y_{\ell 0}(\theta,\varphi)$$

$r > a$

for $r < a$ we use the solution with $r^\ell\, Y_{\ell 0}(\theta,\varphi)$

then on z axis $\phi = \sum b^<_\ell\, r^\ell \left(\dfrac{1-(-1)^\ell}{2}\right)$

$$\Rightarrow b^<_\ell = \frac{2 r^\ell q}{a^{\ell+1}} \left(\frac{1-(-1)^\ell}{2}\right)$$

$$= \frac{q}{a\left(1-\frac{r}{a}\right)} - \frac{q}{a\left(1+\frac{r}{a}\right)}$$

$$= \frac{q}{a-r} - \frac{q}{a+r}$$