

Lecture 1

Maxwell equation

- a) Coulomb's law
- b) Electric field
- c) potential
- d) Gauss law.



Maxwell eqs.

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{j} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t}$$

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}$$

The goal of this class is to understand these equations.

In covariant form they are very simple $\partial_\mu F_{\mu\nu} = j_\nu$
↑ ↑ ↙
4 derivative em field tensor 4 current.

ME are relativistically correct but not quantum mechanically.

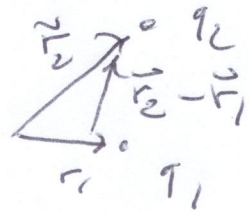
The quantum theory is Quantum Electro Dynamics QED.

The Maxwell equations simplify when the time derivatives vanish.

Then the equations for \vec{E} and \vec{B} decouple, and we have electrostatics and magnetostatics.

a) For electrostatics, all you need is Coulomb's law

$$\vec{F}_{21} = q_1 q_2 \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|^3}$$



$$- \vec{F}_{12} = -\vec{F}_{21}$$

- forces are additive

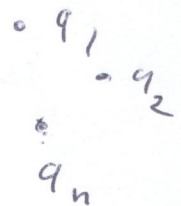
in electrostatics you can solve all problems using Coulomb's law only.

b) Electric field

This is the force per unit charge

$$\vec{E} = \frac{\vec{F}}{q}$$

$$\vec{E}(\vec{r}) = \sum_i q_i \frac{(\vec{r} - \vec{r}_i)}{|\vec{r} - \vec{r}_i|^3}$$



charge density $\rho(\vec{r}) = \sum_i q_i \delta^3(\vec{r} - \vec{r}_i)$

$$\Rightarrow \vec{E}(\vec{r}) = \int d^3 r' \rho(\vec{r}') \frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}$$

c) Potential

3

$$\frac{\vec{F} - \vec{r}_c}{|\vec{r} - \vec{r}_c|^3} = -\vec{\nabla} \frac{1}{|\vec{r} - \vec{r}_c|}$$

$$= -\partial_x \times \frac{1}{\sqrt{(x_1 - x_{1c})^2 + (x_2 - x_{2c})^2 + (x_3 - x_{3c})^2}}$$

$$\Rightarrow \vec{E} = \vec{\nabla} \cdot \underbrace{\sum_i \frac{q_i}{|\vec{r} - \vec{r}_i|}}_{\text{potential } \phi(\vec{r})}$$

potential $\phi(\vec{r})$

$$\phi(\vec{r}) = \sum_i \frac{q_i}{|\vec{r} - \vec{r}_i|} = \int \rho(r') \frac{1}{|\vec{r} - \vec{r}'|}$$

Since $\vec{E} = \vec{\nabla} \cdot \phi$

we have that $\vec{\nabla} \times \vec{E} = \vec{\nabla} \times \vec{\nabla} \phi = 0$

So this ME follows from Coulomb's law

We can also invert the equation

$$\phi(\vec{r}) = - \int_{\vec{r}_0}^{\vec{r}} \vec{E}(\vec{s}) \cdot d\vec{s}$$



to be unique we need

$$- \int_1^{\vec{r}} \vec{E} \cdot d\vec{s} = \int_2^{\vec{r}} \vec{E} \cdot d\vec{s}$$

This is the case of

$$\oint \vec{E} \cdot d\vec{s} = 0$$

Stokes theorem

$$\oint_{\partial C} \vec{E} \cdot d\vec{s} = \int_C \underbrace{\vec{\nabla} \times \vec{E}}_0 \cdot d\vec{A}$$

$$= 0$$

d) Gauss Law

We have that $\vec{E} = \int \frac{\rho(\vec{r}') (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}$

now we want to invert this equation and express ρ in \vec{E} .

We first prove

$$\vec{\nabla}_{\vec{r}} \cdot \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} = 4\pi \delta^3(\vec{r} - \vec{r}')$$

$$= \vec{\nabla}_{\vec{r}} \cdot \vec{\nabla}_{\vec{r}} \frac{1}{|\vec{r} - \vec{r}'|}$$

$\vec{\nabla}_{\vec{r}}^2$ is known as the Laplacian

$$\vec{\nabla}_{\vec{r}}^2 = \partial_x^2 + \partial_y^2 + \partial_z^2$$

\vec{r}' does not matter, so we choose $\vec{r}' = 0$

We have to prove that

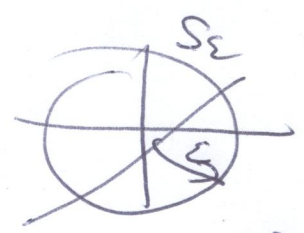
$$\vec{\nabla}_r^2 \frac{1}{|\vec{r}|} = -4\pi \delta^3(\vec{r})$$

OK if $\vec{r} \neq 0$

$$\begin{aligned} \frac{\partial}{\partial x} \frac{1}{\sqrt{x^2+y^2+z^2}} &= \frac{\partial}{\partial x} \frac{x}{(x^2+y^2+z^2)^{3/2}} \\ &= \frac{-1}{(x^2+y^2+z^2)^{3/2}} + \frac{3x}{(x^2+y^2+z^2)^{5/2}} \end{aligned}$$

$$\Rightarrow (\partial_x^2 + \partial_y^2 + \partial_z^2) \frac{1}{\sqrt{x^2+y^2+z^2}} = 0$$

To prove that is is also correct around zero we put a small sphere around zero and integrate about it



$$\int_{S_\epsilon} d^3r \vec{\nabla}_r \cdot \vec{\nabla}_r \frac{1}{|\vec{r}|} = \int_{\partial S_\epsilon} d\vec{a} \cdot \vec{\nabla}_r \frac{1}{|\vec{r}|}$$

Gauss theorem

$$d\vec{a} = r^2 \hat{r} d\Omega = \hat{r} r^2 \sin\theta d\theta d\phi$$

$$= \int_{\partial S_\epsilon} r^2 d\Omega \cdot \frac{-1}{r^2} = - \int d\Omega = -4\pi$$

The rhs just gives $4\pi \Rightarrow$ q.e.d.

We have shown that

(6)

$$\vec{\nabla} \cdot \frac{1}{|\vec{r}-\vec{r}'|} = -4\pi \delta^3(\vec{r}-\vec{r}')$$

$$\begin{aligned} \Rightarrow \vec{\nabla} \cdot \vec{E} &= \int d^3\vec{r}' \left(\vec{\nabla} \cdot \frac{1}{|\vec{r}-\vec{r}'|} \right) \rho(\vec{r}') \\ &= \int d^3\vec{r}' \rho(\vec{r}') (-4\pi) \delta^3(\vec{r}-\vec{r}') \\ &= -4\pi \rho(\vec{r}) \end{aligned}$$

Gauss Law