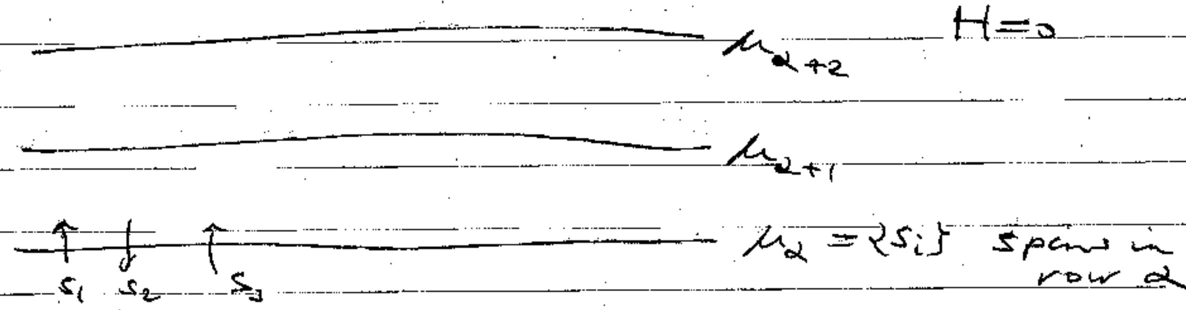
11) Onsager solution of the 2d Ising model



$$Z = \mathrm{Tr}\, e^{K \sum_{\langle ij \rangle} s_i s_j}$$

$$= \sum e^{K \sum_{\alpha=1}^{N} E(\mu_\alpha, \mu_{\alpha+1}) + K \sum_{\alpha=1}^{n} E(\mu_\alpha)}$$

$$E(\mu_\alpha, \mu_{\alpha+1}) = \sum_{\substack{i \in \mu_\alpha \\ j \in \mu_{\alpha+1} \\ ij\ \text{n.n.}}} S_i S_j \qquad E(\mu_\alpha) = \sum_{\substack{i \in \mu_\alpha \\ j \in \mu_\alpha \\ ij\ \text{n.n.}}} S_i S_j$$

$$\Rightarrow Z = \sum_{\mu_1} \cdots \sum_{\mu_n} e^{K(E(\mu_1,\mu_2) + E(\mu_1))} \times e^{K(E(\mu_2,\mu_3) + E(\mu_2))} \times \cdots$$

$$\Rightarrow Z = \mathrm{Tr}\, T^n$$

$$T_{\mu\mu'} = e^{K(E(\mu,\mu') + E(\mu'))}$$

T is a $2^n \times 2^n$ matrix, $\mu = \{s_i\}$ has 2^n elements

$$\Rightarrow Z = \lambda_{max}^n \quad \text{for } n \to \infty$$

Task: Find λ_{max}

a)

$$\langle s_1 \dots s_n | e^{KE(\mu,\mu')} | s_1' \dots s_n' \rangle = V_1 \qquad \langle s_1' \dots s_n' | e^{KE(\mu')} | s_1' \dots s_n' \rangle = V_2$$

$$T = e^{KE(\mu,\mu')} e^{KE(\mu')}$$

$$\langle S_1 \dots S_n | T | S_1' \dots S_n' \rangle = V_1 V_2 \qquad (51)$$

T matrices

$$\langle S_1 \dots S_n | V_1 | S_1' \dots S_n' \rangle = \prod_K e^{K S_K S_K'}$$

$$\langle S_1 \dots S_n | V_2 | S_1' \dots S_n' \rangle = \delta_{s_1 s_1'} \dots \delta_{s_n s_n'} \, e^{K S_K S_{K+1}}$$

V_1

only spins in one row interact via V_1

$\Rightarrow V_1$ is the direct product of 1d transfer matrices

$$V_1 = \underbrace{a \times a \dots \times a}_{n \text{ times}}$$

$$a' = \begin{pmatrix} e^{K} & e^{-K} \\ e^{-K} & e^{K} \end{pmatrix}$$

$$= e^{K} + e^{-K} \sigma_x$$

$$= \sqrt{e^{2K} - e^{-2K}} \Big(\underbrace{\frac{e^{K}}{\sqrt{e^{2K} - e^{-2K}}}}_{\cosh \theta} + \underbrace{\frac{e^{-K} \sigma_x}{\sqrt{e^{2K} - e^{-2K}}}}_{\rightarrow \sinh \theta} \Big)$$

$$\Rightarrow \quad a = \sqrt{e^{2K} - e^{-2K}} \, e^{\theta \sigma_x}$$

$$\sigma_\alpha^x = 1 \times 1 \times \dots \times \underset{\uparrow \text{position}}{\sigma_\alpha^x} \times 1 \dots \times 1$$

$$\sigma_\alpha^z = 1 \times 1 \times \quad \times \sigma_\alpha^z \times 1 \dots \times 1$$

$$\Rightarrow V_1 = \left(\sqrt{e^{2K} - e^{-2K}} \right)^n \left(e^{\theta \sigma_1^x} \dots e^{\theta \sigma_n^x} \right)$$

$$= \left(\sqrt{e^{2K} - e^{-2K}} \right)^n e^{\theta (\sigma_1^x + \dots + \sigma_n^x)}$$

$$V_2 = \prod_\alpha e^{K \sigma_\alpha^z \sigma_{\alpha+1}^z}$$

$$\sigma_\alpha^z |S_1 \ldots S_\alpha \ldots S_n\rangle = S_\alpha |S_1 \ldots S_\alpha \ldots S_n\rangle$$

$$\sigma_{n+1}^z = \sigma_1^z$$

Next we rotate so that spins $\parallel$ x-axis

$$\sigma_x \longrightarrow \sigma_z$$
$$\sigma_z \longrightarrow -\sigma_x$$

$$\Rightarrow V_1 = \left(\sqrt{e^{2K} - e^{-2K}}\right)^n e^{\Theta(\sigma_1^z + \ldots + \sigma_n^z)}$$

$$V_2 = \prod_\alpha e^{K \sigma_\alpha^x \sigma_{\alpha+1}^x}$$

b) <u>New operators</u>

$$\sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$
$$\sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$c_m = (-1)^{\sum_{j<m} n_j} \sigma_m^- \qquad n_j = \sigma_j^+ \sigma_j^- = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$c_m^* = (-1)^{\sum_{j<m} n_j} \sigma_m^+$$

$$c_m c_{m'} = (-1)^{\sum_{j<m} n_j} \sigma_m^- (-1)^{\sum_{j<m'} n_j} \sigma_{m'}^- \qquad \text{take } m \gtrless m'$$

$$= (-1)^{\sum_{j<m} n_j} (-1)^{\sum_{j<m'} n_j + 1} \sigma_m^- \sigma_{m'}^-$$

$m' > m \Rightarrow$ commute

$$= (-1)\, c_{m'} c_m \qquad \Rightarrow \qquad c_m^2 = 0$$

$\{C_m, C^*_{m'}\} = \delta_{mm'}$ Prove this as an exercise

- The C_m do not have a simple direct product structure

$$\Rightarrow \sigma^x_m \sigma^x_{m+1} = (\sigma_m^+ + \sigma_m^-)(\sigma_{m+1}^+ + \sigma_{m+1}^-)$$
$$= \sigma_m^x (-1)^{\sum_{j<m+1} n_j} (C_{m+1} + C^*_{m+1})$$

$$\sigma_m^x (-1)^{n_m} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -1 & \\ & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\sigma_{y\,m}$$
$$= \sigma_m^+ - \sigma_m^-$$

$$V_2 = e^{K \sum_m \sigma_m^x \sigma_{m+1}^x} = e^{K \sum_m (\sigma_m^+ - \sigma_m^-)(-1)^{\sum_{j<m} n_j}(C_{m+1} + C^*_{m+1})}$$
$$= e^{K \sum_m (-1)^{\sum_{j<m} n_j} (-1)^{\sum_{j<m} n_j} (C_m^* - C_m)(C_{m+1} + C^*_{m+1})}$$

$$V_1 = (e^{2K} - e^{-2K})^{\frac{N}{2}} \, e^{-\theta \sum_m (2n_m - 1)}$$

$$\sigma_{mz} = 2\sigma_m^+ \sigma_m^- - 1 = 2n_m - 1$$

boundary conditions:

$$C_{n+1} = (-1)^{\sum_{j<n+1} n_j} \sigma^-_{n+1} \qquad \sigma_n^+ \sigma_{n+1}^+ = \sigma_n^+ \sigma_1^+$$
$$\sigma^-_{n+1} = \sigma_1^-$$

$$C_n C_1 = (-1)^{\sum_{j<n} n_j} \sigma_n^- \sigma_1^-$$
$$C_n C_{n+1} = (-1)^{\sum_{j<n} n_j} \sigma_n^- (-1)^{\sum_{j<n+1} n_j} \sigma^-_{n+1}$$
$$= -(-1)^{\sum_{j<n+1} n_j} (-1)^{\sum_{j<n} n_j} \sigma_n^- \sigma_1^-$$

$$\Rightarrow C_{n+1} = -\sigma_1^- \text{ if } \sum_j n_j = \text{even}$$
$$C_{n+1} = \sigma_1^- \text{ if } \sum_j n_j = \text{odd}$$