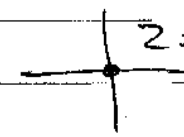
9a) Low Temperature expansion

$$Z = \sum e^{K \sum_{\langle ij \rangle} S_i S_j}$$



$z = 4$

$T \to 0$ then $K \to \infty$

$$Z = \underbrace{e^{KN\frac{z}{2}}}_{\text{all spins up}} + \underbrace{N e^{KN\frac{z}{2} - 2Kz}}_{\text{all spins up except one}} + \dots$$

higher order terms get more and more complicated because we have to distinguish neighboring spins from others

9b) High-T expansion

$$Z = \sum_{\{S_i\}} \prod_{\langle ij \rangle} \underbrace{\left(1 + S_i S_j \frac{\sinh K}{\cosh K}\right) \cosh K}_{\begin{subarray}{l} = e^{K} \text{ if } S_i = S_j \\ = e^{-K} \text{ if } S_i = -S_j \end{subarray}}$$

$$= \sum_{\{S_i\}} \prod_{\langle ij \rangle} \left(1 + S_i S_j \tanh K\right) (\cosh K)^{\frac{zN}{2}}$$

9c) <u>Mean Field theory</u>

$$Z = \sum_{\{S_i\}} e^{\sum_{\langle ij\rangle} K S_i S_j + \sum_i h S_i}$$

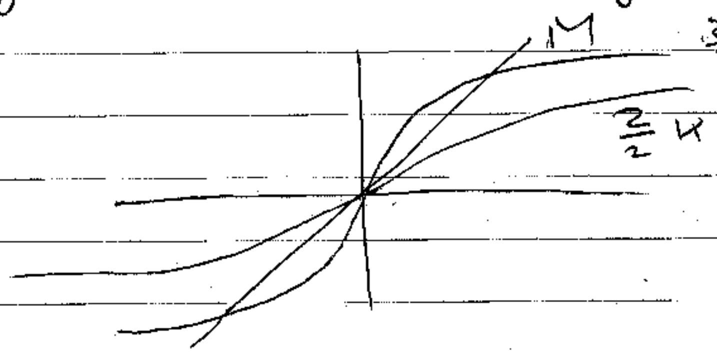
$$Z_{MFT} = \sum_{\{S_i\}} e^{K \sum_i S_i \frac{z}{2} \underset{M}{\underset{\parallel}{\langle S_j \rangle}} + \sum_i h S_i}$$

$$\Rightarrow Z_{MFT} = \sum_{\{S_i\}} e^{\sum_i S_i (M \frac{z}{2} K + h)} = 2^N \cosh^N(h + \tfrac{z}{2} K M)$$

magnetization $M = \frac{1}{N} \partial_h \log Z = \tanh(h + \frac{z}{2} K M)$

Do we have spontaneous magnetization? $(h=0)$

graphical solution of $M = \tanh \frac{z}{2} K M$



$\Rightarrow K_c = \frac{2}{z}$

$$M = \frac{Kz}{2} M - \frac{1}{3}(K \tfrac{z}{2} M)^3$$

$$\frac{Kz}{2} = 1 + \delta \quad \Rightarrow \quad M = M + \delta M - \tfrac{1}{3} M^3 (1+\delta)^3$$

$$\Rightarrow \delta = \tfrac{1}{3} M^2 + \mathcal{O}(M^3)$$

$$\Rightarrow M \sim \delta^{\frac{1}{2}} \quad \Rightarrow \beta = \tfrac{1}{2}$$

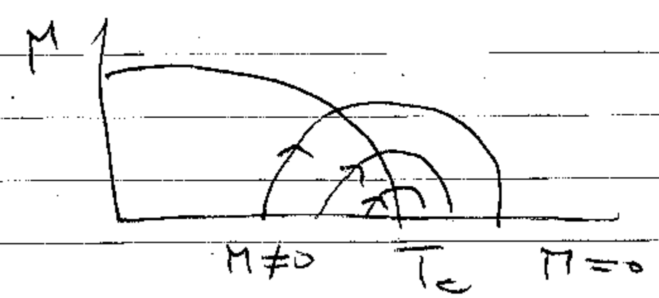
1° Kramers-Wannier duality

Theorem: The 2d Ising model satisfies the relation

$$Z(N,K) = (\sinh 2K^*)^{-N} Z(N,K^*)$$

with $\sinh 2K^* \sinh 2K = 1$

Consequence The partition function of a low temperature phase is mapped onto the partition function of a high temperature phase



This is also the case close to T_c

$$\Rightarrow \sinh 2K_c = \frac{1}{\sinh 2K_c} \Rightarrow e^{2K_c} = \sqrt{2} + 1$$

This determines the critical temperature of the Ising model

The relation can be rewritten as

$$\frac{Z(N,K)}{(\sinh 2K)^{N/2}} = \frac{Z(N,K^*)}{(\sinh 2K^*)^{N/2}}$$

Proof of the duality relation

$$Z(N,K) = \sum_{\{S_i\}} e^{K\sum_{\langle ij\rangle} S_i S_j}$$

$$= \sum_{\{S_i\}}\prod_{\langle ij\rangle}\left(\underbrace{\cosh K}_{C_0} + S_iS_j \underbrace{\sinh K}_{C_1}\right)$$

$$= \sum_{\{S_i\}}\prod_{\langle ij\rangle}\sum_{k=0}^{1} C_k (S_iS_j)^k$$

$$= \sum_{\{S_i\}}\sum_{k_{ij}=0}^{1}\prod_{\langle ij\rangle} C_{k_{ij}} (S_iS_j)^{k_{ij}}$$

$$= \sum_{\{S_i\}}\sum_{k_{ij}=0}^{1}\prod_{\langle ij\rangle} C_{k_{ij}} \underbrace{\prod_i \prod_{j\ \text{n.n. of } i} S_i^{k_{ij}}}_{\prod_i S_i^{\sum_{j n i} k_{ji}}}$$

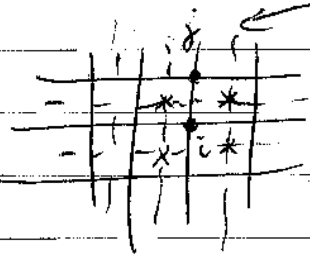
sum of product = product of sums

$$= \sum_{k_{ij}=0}^{1}\prod_{\langle ij\rangle} C_{k_{ij}} \prod_i \left(\underbrace{1 + (-1)^{\sum_{j n i} K_{ij}}}_{\text{only nonzero}}\right)$$

$$= 2^N \sum_{k_{ij}=0}^{1}{}' \prod_{ij} C_{k_{ij}} \qquad \text{if } \sum_{j n i} K_{ij} = \text{even}$$

prime indicates that $\sum_{j n i} K_{ij} =$ even

To proceed we need to introduce the dual lattice



← dual lattice (denoted by $*$)

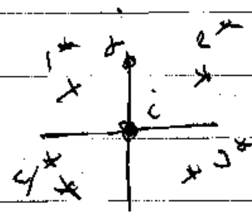
ij is dual link

trick: Instead of summing $\sum'_{K_{ij}}$ we sum over the dual lattice $\sum_{\sigma_i^*}$

with $K_{ij} = \frac{1}{2}(1 - \sigma_i^* \sigma_j^*)$

$$\Rightarrow \sum_{j\sim i} K_{ij} = \sum_{j\sim i} \left(\frac{1}{2} - \frac{1}{2}\sigma_i^* \sigma_j^*\right)$$

$$= 2 - \frac{1}{2}\left(\sigma_{1^*}\sigma_{2^*} + \sigma_{2^*}\sigma_{3^*} + \sigma_{3^*}\sigma_{4^*} + \sigma_{4^*}\sigma_{1^*}\right)$$



$$= 2 - \frac{1}{2}\left(\sigma_{1^*} + \sigma_{3^*}\right)\left(\sigma_{2^*} + \sigma_{4^*}\right)$$

$$2, 0, 0, -2 \qquad 2, 0, 0, -2$$

degeneracies

0	2
2	12
4	2

degeneracies for $\sum_{j\sim i} K_{ij} =$

0	1
2	6
4	1

$$\cosh K^* = e^{2K} \sinh K^*$$

$$\Rightarrow \sinh 2K = \frac{1}{2}\left(e^{2K} - e^{-2K}\right) = \frac{1}{2}\left(\frac{\cosh K^*}{\sinh K^*} - \frac{\sinh K^*}{\cosh K^*}\right)$$

$$= \frac{1}{2\cosh K^* \sinh K^*}$$

$$= \frac{1}{\sinh 2K^*}$$

$$\Rightarrow Z = \left(\frac{1}{\sinh 2K^*}\right)^N \underbrace{\sum_{\{\sigma_i\}} e^{\sum_{ij} K^* \sigma_i \sigma_j}}_{Z(N, K^*)}$$

$$\left(\frac{2^{-N} e^{2KN}}{(\cosh K^*)^{2N}}\right) = \frac{2^{-N}}{\cosh^N K^* \sinh^N K^*}$$

$$= \frac{1}{\sinh^N 2K^*}$$