8) Ising model

Definition $\mathcal{H} = -J \sum_{\langle ij \rangle} S_i S_j - H \sum_i S_i$

$J > 0$ ↑ nearest neighbors

$h = \beta H \qquad K = \beta J$

8a) Solution in 1d

$$Z = \sum_{\{S_i\}} e^{h \sum_{i=1}^{N} S_i + K \sum_{i=1}^{N} S_i S_{i+1}}$$

- free boundary conditions $\qquad S_{N+1}$ is free

sum over links $\quad \eta_i = S_i S_{i+1} = \pm 1$

For $H = 0$

$$Z = \sum_{S_1 = -1}^{1} \prod_{i=1}^{N} \sum_{\eta_i = -1}^{1} e^{K \eta_i}$$

$$\Rightarrow Z = 2 \, (2 \cosh K)^N$$

- periodic boundary conditions

$S_{N+1} = S_1$

$$\sum_{\{S_i\}} = \sum_{S_1 = -1}^{1} \sum_{\eta_1 = -1}^{1} \cdots \sum_{\eta_{N-1} = -1}^{1}$$

$S_N = \eta_{N-1} S_{N-1}$
$= \eta_{N-1} \cdots \eta_1 S_1$

$$\Rightarrow \quad Z = \sum_{s_1} \sum_{\eta_1} \cdots \sum_{\eta_{N-1}} e^{K(\eta_1 + \cdots + \eta_{N-1}) + K\eta_1 \cdots \eta_{N-1}}$$

gives factor 2

$$= 2 \sum_{\eta_1} \cdots \sum_{\eta_{N-1}} \sum_{p=0}^{\infty} \frac{1}{p!} K^p \eta_1^p \cdots \eta_{N-1}^p \, e^{K(\eta_1 + \cdots + \eta_{N-1})}$$

$$= 2 \sum_{p=0}^{\infty} \frac{1}{p!} K^p \prod_{i=1}^{N-1} \underbrace{\sum_{\eta_i = -1}^{1} \eta_i^p e^{K\eta}}_{(-1)^p e^{-K} + e^{K}}$$

$$= 2 \sum_{p=0}^{\infty} \frac{1}{p!} K^p \left((-1)^p e^{-K} + e^{K} \right)^{N-1}$$

$$= 2 \sum_{p = \text{even}} \frac{1}{p!} K^p (2\cosh K)^{N-1} +$$

$$+ 2 \sum_{p \text{ odd}} \frac{1}{p!} K^p (2\sinh K)^{N-1}$$

$$= 2^N \cosh^N K + 2^N \sinh^N K$$

§b) <u>Transfer matrix</u>

$$Z_N = \mathrm{Tr}\; e^{h\sum_{i=1}^{N} s_i + K\sum_{i=1}^{N} s_i s_{i+1}} \qquad s_{N+1} = s_1$$

$$= \sum_{s_1} \cdots \sum_{s_N} e^{\frac{h}{2}(s_1+s_2)+K s_1 s_2}\; e^{\frac{h}{2}(s_2+s_3)+K s_2 s_3}$$

sum over s_2 is matrix multiplication

$$\Rightarrow Z_N = \mathrm{Tr}\; T^N \qquad T \text{ is transfer matrix}$$

$$T = \begin{pmatrix} e^{h+K} & e^{-K} \\ e^{-K} & e^{-h+K} \end{pmatrix}$$

$$\Rightarrow Z_N = \lambda_1^N + \lambda_2^N$$

λ_1 and λ_2 are eigenvalues of T

$$\left(e^{h+K} - \lambda\right)\left(e^{-h+K} - \lambda\right) - e^{-2K} = 0$$

$$\Rightarrow \lambda^2 - \lambda e^{K}\left(e^{h} + e^{-h}\right) + e^{2K} - e^{-2K}$$

$$\Rightarrow \lambda_{1,2} = \frac{e^{K}\left(e^{h}+e^{-h}\right) \pm \sqrt{e^{2K}\left(e^{h}+e^{-h}\right)^2 - 4e^{2K} + 4e^{-2K}}}{2}$$

$$= e^{K}\cosh h \pm \sqrt{e^{2K}\sinh^2 h + e^{-2K}}$$

We use the transfer matrix to calculate the correlation function

$$\langle S_i \rangle = \frac{1}{2} \sum_{\{S_i\}} e^{-\beta H} S_i$$

$$= \frac{1}{2} \operatorname{Tr} T_{S_1 S_2} T_{S_2 S_3} \cdots T_{S_{i-1} S_i} \underset{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}{\underset{\parallel}{S_i}} T_{S_i S_{i+1}}$$

$$= \frac{1}{2} \operatorname{Tr} T^N \sigma_z$$

$$\langle S_i S_j \rangle = \frac{1}{2} \operatorname{Tr} T_{S_1 S_2} \cdots S_{i-1} S_i S_{i+1} \cdots S_{j-1} S_j S_{j+1}$$

$$= \frac{1}{2} \operatorname{Tr} T^{N-j+i} \sigma_z T^{j-i} \sigma_z$$

We calculate the trace by diagonalizing T

$$T = S \Lambda S^{-1} \qquad S = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}$$

$$\Rightarrow \langle S_i \rangle = \frac{1}{2} \operatorname{Tr} \begin{pmatrix} \lambda_1^N & 0 \\ 0 & \lambda_2^N \end{pmatrix} S^{-1} \sigma_z S$$

$\rightarrow \lambda_1^N$ for N

$$= (S^{-1} \sigma_z S)_{11} = \cos^2\varphi - \sin^2\varphi$$

$$S \Lambda S^{-1} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix} \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_1 \cos^2\varphi + \lambda_2 \sin^2\varphi & \\ & \lambda_1 \sin^2\varphi + \lambda_2 \cos^2\varphi \end{pmatrix}$$

$$\Rightarrow (T_{11} - T_{22}) = (\lambda_1 - \lambda_2)(\cos^2\varphi - \sin^2\varphi)$$

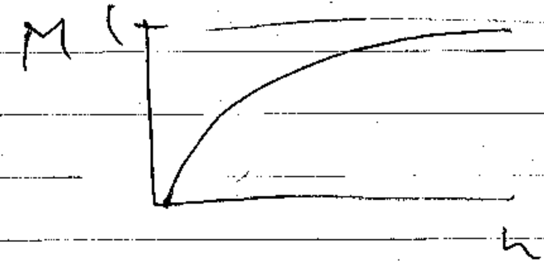
In the thermodynamic limit the partition function is dominated by the largest eigenvalue

$$Z = \lambda_1^N \Rightarrow f = \frac{F}{N} = -k_B T \log \lambda$$

$$K = \frac{J}{k_B T} \Rightarrow f = -J - k_B T \log\left(\cosh h + \sqrt{\sinh^2 h + e^{-4K}}\right)$$

$T \neq 0$ f is a smooth function of T
$\Rightarrow$ no phase transition for $T > 0$

$$M = -\frac{1}{N}\frac{\partial F}{\partial H} = \partial_h \log \lambda_1 = \frac{\sinh h}{\sqrt{\sinh^2 h + e^{-4K}}}$$



(c) Correlation functions

$$G_{ij} = \langle S_i S_j \rangle - \langle S_i \rangle \langle S_j \rangle$$
measures probability that S_i and S_j have the same value.

$$p_{ij} = \langle \tfrac{1}{2}(1 + S_i S_j) \rangle = \tfrac{1}{2} + \tfrac{1}{2}\langle S_i \rangle \langle S_j \rangle + \tfrac{1}{2} G_{ij}$$

$$\Rightarrow \cos^2\varphi - \sin^2\varphi = \frac{(e^h - e^{-h})}{2\sqrt{\sinh^2 h + e^{-4K}}}$$

$$\Rightarrow \langle S_i \rangle = \frac{\sinh}{\sqrt{\sinh^2 h + e^{-4K}}} \quad \text{as found before}$$

<u>Correlation function</u>

$$\langle S_i S_{i+j} \rangle = \frac{1}{\lambda_1^N} \operatorname{Tr} \begin{pmatrix} \lambda_1^{N-j} & \\ & \lambda_2^{N-j} \end{pmatrix} S^{-1}\sigma_z S \begin{pmatrix} \lambda_1^{j} & \\ & \lambda_2^{j} \end{pmatrix} \times (S^{-1}\sigma_z S)$$

$$= \lambda_1^{-j} \left[S^{-1}\sigma_z S \begin{pmatrix} \lambda_1^j & \\ & \lambda_2^j \end{pmatrix} S^{-1}\sigma_z S \right]_{11}$$

$$= \underbrace{\left[(S^{-1}\sigma_z S)_{11}\right]^2}_{\langle S_i \rangle^2} + \lambda_1^{-j}\lambda_2^{j} \left[(S^{-1}\sigma_z S)_{12}\right]^2$$

$$\Rightarrow G_{i,i+j} = \langle S_i S_{j+i} \rangle - \langle S_i \rangle \underbrace{\langle S_{j+i} \rangle}_{= \langle S_i \rangle}$$

$$= \left(\frac{\lambda_2}{\lambda_1}\right)^j \left((S^{-1}\sigma_z S)_{12}\right)^2$$

$$(S^{-1}\sigma_z S)_{12} = 2\cos\varphi\sin\varphi = 2\frac{T_{12}}{\lambda_2 - \lambda_1}$$

$$= \frac{2e^{-2K}}{2\sqrt{\sinh^2 h + e^{-4K}}}$$

$$\Rightarrow G_{i\,i+j} = \frac{e^{-2K}}{\sqrt{\sinh^2 h + e^{-4K}}} \, e^{-j \log \frac{\lambda_1}{\lambda_2}}$$

$$\Rightarrow \text{correlation length} \quad \xi = \frac{1}{\log \frac{\lambda_1}{\lambda_2}}$$