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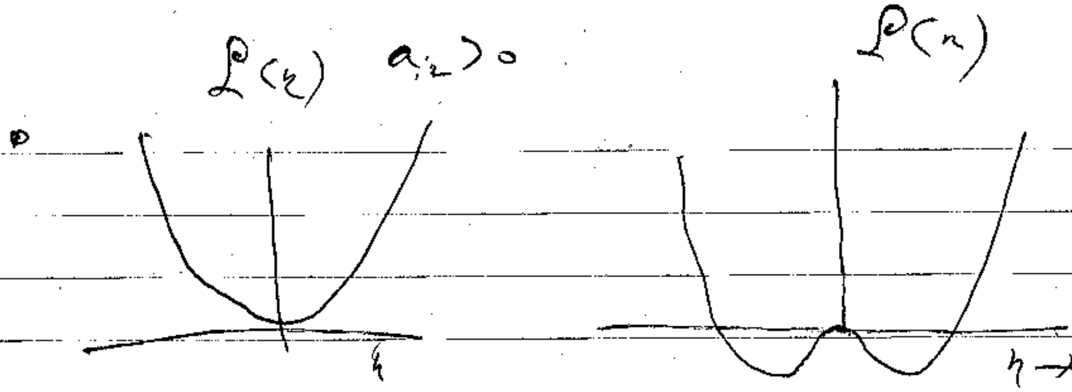


transition happens at $t = 0 \Rightarrow a_2^0 = 0$

Final result: $\mathcal{L} = a t \eta^2 + \frac{1}{2} b \eta^4$ $\quad b > 0$, $a > 0$

magnetic field term in Ising model
$- H \sum S_i = - H N M$
in terms of order parameter $- H \eta V$
breaks Z_2

$$\Rightarrow \mathcal{L} = a t \eta^2 + \frac{1}{2} b \eta^4 - \eta H$$

5c) <u>Critical exponents for LG theory</u>

minimum $\frac{\partial \mathcal{L}}{\partial \eta} = 0 = 2 a t \eta + 2 b \eta^3 - H = 0$

for $H = 0$ $\quad \eta_1 = 0$, $\eta_2 = \sqrt{\frac{-a t}{b}}$

from picture: η_2 is absolute minimum for $t < 0$
η_1 is absolute minimum for $t > 0$

$\eta_2 \sim t^{1/2} \Rightarrow \boxed{\beta = \frac{1}{2}}$

$$\chi = \frac{1}{V}\frac{\partial M}{\partial H} = \frac{1}{V}\frac{\partial \eta}{\partial H}$$

$$2at\frac{\partial \eta}{\partial H} + 2\,3\,b\,\eta^2\frac{\partial \eta}{\partial H} - 1 = 0 \qquad (\eta \neq 0)$$

For $\eta^2 = -\frac{at}{b}$ we have $\frac{\partial \eta}{\partial H} = \frac{-1}{4at}$

$$\Rightarrow \boxed{\gamma = 1}$$

at $t = 0$ $\quad \eta^3 = \frac{H}{2b} \Rightarrow \eta \sim H^{1/3}$

$$\Rightarrow \boxed{\delta = 3}$$

specific heat; $\quad c = -T\frac{\partial^2 \mathcal{L}}{\partial T^2}$

$$\mathcal{L}_{minimum} = at\left(-\frac{at}{b} + \frac{1}{2}b\,\frac{a^2t^2}{b^2}\right) = -\frac{a^2t^2}{b}$$

$t < 0 \Rightarrow \qquad c = \frac{a^2}{bT_c} \Rightarrow \alpha = 0$

$t > 0$ then $\mathcal{L} = 0 \Rightarrow c = 0 \Rightarrow \alpha' = 0$

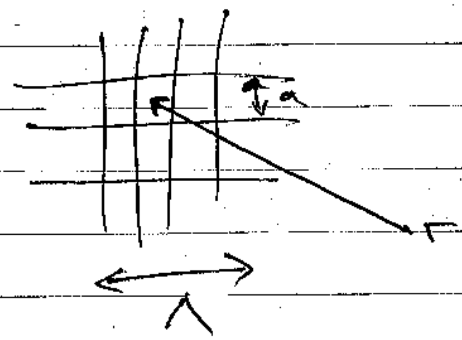
<u>Relations</u> $\qquad \alpha + 2\beta + \gamma = 2$

$0 + 2\frac{1}{2} + 1 = 2 \quad$ o.k

$$\beta(\delta - 1) = \gamma$$

$\frac{1}{2}(3-1) = 1 \quad$ o.k

5d) Inhomogeneous systems



divide space into cells of volume Λ^d

Magnetization: $\vec{M}(\vec{r}) = \frac{1}{N_\Lambda} \sum_{i \in \Lambda(\vec{r})} \langle S_i \rangle$

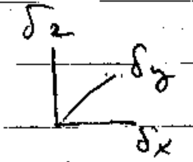
coarse graining

Magnetization of neighboring cells tend to be parallel. This is achieved by the term

$$\gamma \sum_{\vec{r},\vec{\delta}} \left(M_\Lambda(\vec{r}) - M_\Lambda(\vec{r}+\vec{\delta}) \right)^2$$

$$\downarrow$$

$$M_\Lambda(\vec{r}) + \vec{\delta} \cdot \vec{\nabla} M_\Lambda(\vec{r})$$



$\delta = \begin{pmatrix} \delta_x \\ \delta_y \\ \delta_z \end{pmatrix}$

$$\sim \Lambda^{2-d} \gamma \int d^d r \left(\vec{\nabla} M_\Lambda(\vec{r}) \right)^2$$

$|\delta_x| = \Lambda$

Full LG functional

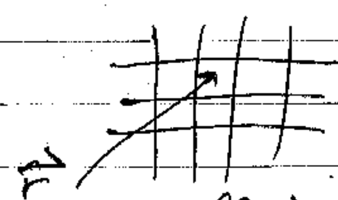
$$\mathcal{L} = \frac{\gamma}{2} (\nabla \eta)^2 + a t \eta^2(r) + \frac{1}{2} b \eta^4(r) - H(r) \eta(r)$$

5e) Meaning of LG functional

$$Z = \mathrm{Tr}\; e^{-\beta H}$$

$$= \sum_{\substack{\text{Coarse} \\ \text{grained variable}}} \; \sum_{\text{microscopic variable}} e^{-\beta H}$$

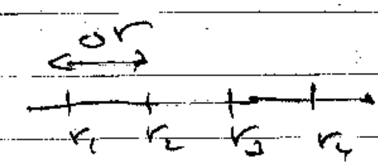
$$= \sum_{\eta(r)} \; \sum_{s_i \in \Delta V} \delta\left(\frac{1}{N_0} \sum_i s_i - \eta(r) \right)$$



cell of N_0 spins
in volume ΔV
$\vec{\eta}(\vec{r})$ is average
magnetization

Landau partition function

$$Z = \sum_{\eta(r)} e^{-\beta L} = \int d\eta(r_1) \cdots d\eta(r_n)\, e^{-\beta L}$$



$\Delta r \to 0$ (continuum limit)

We obtain a functional integral

$$\delta(n_x - n_x') = \frac{2\pi}{L}\,\delta\left(\frac{2\pi}{L}n_x - \frac{2\pi}{L}n_x'\right) = \frac{2\pi}{L}\,\delta(k_x - k_x')$$

$$\Rightarrow \int d^d r\, e^{i(k-k')r} = (2\pi)^d\, \delta^d(k-k')$$

5.4) Response functions

$$\langle m(r)\rangle = -\frac{\delta F}{\delta H(r)}$$

$\uparrow$
order parameter

Susceptibility: $\chi(r,r') = \frac{\delta\langle m(r)\rangle}{\delta H(r')} = -\frac{\delta^2 F}{\delta \ldots}$

$$F = -kT \log Z$$

$$\Rightarrow \chi(r,r') = kT\left(\frac{1}{Z}\frac{\delta^2 Z}{\delta H(r)\,\delta H(r')} - \frac{1}{Z}\frac{\delta Z}{\delta H(r)}\frac{1}{Z}\frac{\delta Z}{\delta H(r')}\right)$$

$$= \frac{kT}{(kT)^2}\left(\langle m(r)\, m(r')\rangle - \langle m(r)\rangle\langle m(r')\rangle\right)$$

$$\Rightarrow \chi(r,r') = \frac{1}{kT}\,\Gamma(r,r') = \beta\,\Gamma(r,r')$$

translational invariance $\Rightarrow$ $\chi(\vec{r},\vec{r}\,') = \chi(\vec{r}-\vec{r}\,')$

5i) The two-point correlation function

$$L = \int d^d r \left(\frac{\gamma}{2}(\nabla \eta)^2 + at\eta^2 + \frac{1}{2} b \eta^4 - H(r)\,\eta(r) \right)$$

$$\frac{\delta L}{\delta \eta(r)} = 0 \Rightarrow -\gamma \nabla^2 \eta + 2at\eta + 2b\eta^3 - H(r) = 0$$

solution $\langle \eta(r) \rangle$

differentiate w.r.t. $H(r')$

susceptibility $\chi = \frac{\delta \langle \eta(r) \rangle}{\delta H(r')}$

$$\Rightarrow -\gamma \nabla^2 \chi + 2at\chi + 6b\eta^2 \chi - \delta^d(r-r') = 0$$

$$\eta^2 = -\frac{at}{b}$$

$$\chi = \chi(r-r')$$

$$\Rightarrow \left(-\nabla^2 + \frac{1}{\xi^2} \right) \Gamma(r) = \frac{kT}{\gamma}\,\delta(r)$$

$$\xi = \left(-\frac{\gamma}{4at} \right)^{1/2} \qquad t > 0$$

$$t > 0 \Rightarrow \eta = 0 \Rightarrow \left(-\nabla^2 + \frac{1}{\xi^2} \right) \Gamma(r) = \frac{kT}{\gamma}\,\delta(r)$$

$$\xi = \left(\frac{\gamma}{2at} \right)^{\frac{1}{2}}$$

we have to solve this diff. eq.

In Fourier space $\Gamma(r) = \frac{1}{(2\pi)^d} \int d^d k\, \Gamma(k) e^{ikr}$

$$\Rightarrow \frac{1}{(2\pi)^d} \int d^d K \; \Gamma(K) \left(K^2 + \frac{1}{\xi^2} \right) e^{iKr} = \frac{1}{\beta\gamma} f(r)$$

$$= \frac{1}{\beta\gamma} \int \frac{d^d K}{(2\pi)^d} e^{iKr}$$

$$\Rightarrow \Gamma(K) = \frac{1/\beta\gamma}{K^2 + \frac{1}{\xi^2}}$$

$$\Gamma(r) = \frac{1}{(2\pi)^d} \int d^d K \frac{1/\beta\gamma}{K^2 + \frac{1}{\xi^2}} e^{iKr}$$

$rK = K'$

$$= \frac{1}{(2\pi)^d} \frac{1}{r^{d-2}} \int d^d K' \frac{1/\beta\gamma}{K'^2 + \left(\frac{r}{\xi}\right)^2} e^{iK'\hat{r}}$$

$$= \frac{1}{r^{d-2}} \frac{1}{(2\pi)^d} \int_0^\infty K^{d-1} dK \int_0^\pi d\theta \sin^{d-2}\theta \; \frac{1/\beta\gamma}{K^2 + \left(\frac{r}{\xi}\right)^2} e^{iK\cos\theta}$$

$$= \frac{1}{r^{d-2}} \frac{1}{(2\pi)^d} \int_0^\infty K^{d-1} dK \underbrace{\int d\theta \sin^{d-2}\theta \; \frac{1/\beta\gamma}{K^2 + \left(\frac{r}{\xi}\right)^2} \cos(K\cos\theta)}_{\left(\frac{2}{K}\right)^{\frac{d-2}{2}} \Gamma\left(\frac{d-1}{2}\right) J_{\frac{d-2}{2}}(K) \frac{1}{K^2 + \frac{r^2}{\xi^2}}}$$

$$= \frac{2^{\frac{d-2}{2}}}{r^{d-2}} \Gamma\left(\frac{d-1}{2}\right) \int_0^\infty dK \; K^{d/2} J_{\frac{d-2}{2}}(K) \frac{1/\beta\gamma}{K^2 + \left(\frac{r}{\xi}\right)^2}$$

$$= \frac{2^{\frac{d-2}{2}}}{r^{d-2}} \Gamma\left(\frac{d-1}{2}\right) \left(\frac{r}{\xi}\right)^{\frac{d}{2}-1} K_{\frac{d}{2}-1}\left(\frac{r}{\xi}\right)$$

$$\frac{r}{\xi} \to \infty \quad \Rightarrow \quad K_{\frac{d}{2}-1}\left(\frac{r}{\xi}\right) = \sqrt{\frac{\pi}{2 r/\xi}} \; e^{-r/\xi}$$

(28)

$$\Rightarrow \Gamma(r) = c\,\frac{1}{r^{d-2}}\left(\frac{r}{\xi}\right)^{\frac{d}{2}-1}\left(\frac{r}{\xi}\right)^{-\frac{1}{2}} e^{-r/\xi}$$

$$= \frac{1}{r^{d-2}}\left(\frac{r}{\xi}\right)^{\frac{d-2-1}{2}} e^{-r/\xi} \propto \Rightarrow$$

$$\sim r^{-\frac{d}{2}+\frac{1}{2}} e^{-r/\xi}$$ ξ is correlation length

$T \to T_c$ $\quad \xi \to \infty \quad \Rightarrow \frac{r}{\xi} \to 0$

$$K_{\frac{d-2}{2}}\left(\frac{r}{\xi}\right) \sim \left(\frac{\xi}{r}\right)^{\frac{d-2}{2}}$$

$$\Rightarrow \Gamma(r) \sim \frac{1}{r^{d-2}}\left(\frac{r}{\xi}\right)^{\frac{d}{2}-1}\left(\frac{\xi}{r}\right)^{\frac{d}{2}-1}$$

$$\sim \frac{1}{r^{d-2}} \Rightarrow \boxed{\eta = 0}$$

$$\xi \sim \frac{1}{t^{1/2}} \Rightarrow \boxed{\nu = \frac{1}{2}}$$