Alternative relations

$$\left.\begin{aligned}\beta\delta &= \nu(d+2-\eta)/2 \\ \beta &= \nu(d-2+\eta)/2\end{aligned}\right\} \Rightarrow \beta\delta - \beta = (2-\eta)\nu$$

$$\Rightarrow \beta(\delta-1) = \gamma$$

Widom

$$2\beta + \alpha = 2 - 2\nu + \nu\eta = 2 - \gamma$$

$$\Rightarrow \quad \alpha + 2\beta + \gamma = 2 \qquad \text{Rushbrook}$$

4c) Explanation

Near the critical point there is only one length scale, the correlation length ξ

$$\xi = \xi(h, t) \qquad h = \frac{H}{kT}$$

central idea: physics near the critical point is scale invariant, physics stays the same if we change the length scale

$$\xi'^{d} \, f(h', t') = \xi^{d} f(h, t)$$

↑ has dimension length^{-d}

$$h' = \left(\frac{\xi'}{\xi}\right)^{-\Delta_h} h \qquad t' = \left(\frac{\xi'}{\xi}\right)^{-\Delta_t} t$$

$$\Rightarrow \quad \xi'^{d} \, f\left(\left(\frac{\xi'}{\xi}\right)^{-\Delta_h} h, \left(\frac{\xi}{\xi}\right)^{-\Delta_t} t\right) = \xi^{d} f(h, t)$$

By changing the length scale we can always transform to $t' = 1$

$$\Rightarrow \left(\frac{\xi'}{\xi}\right)^{-D_t} t = 1$$

$$\Rightarrow f(h,t) = t^{\frac{d}{D_t}} f\left(t^{-\frac{D_h}{D_t}} h, t=1\right)$$

at $t=1$ f is nonsingular

$$\varepsilon \sim \partial_t^2 f(h, t=0) \sim t^{\frac{d}{D_t}-2}$$

$$\Rightarrow \alpha = 2 - \frac{d}{D_t}$$

$$\frac{M}{V} = -\frac{\partial F}{\partial H}\Big|_{h=0} \sim t^{\frac{d}{D_t} - \frac{D_h}{D_t}} \Rightarrow \beta = \frac{d - D_h}{D_t}$$

$$\chi = -\frac{1}{V}\frac{\partial M}{\partial H} \sim \partial_h^2 f\Big|_{h=0} \sim t^{\frac{d}{D_t} - \frac{2D_h}{D_t}}$$

$$\Rightarrow \gamma = \frac{2D_h - d}{D_t}$$

$$f \sim \xi^{-d} \sim t^{\nu d} \Rightarrow \nu = \frac{1}{D_t}$$

$$\chi \sim \int d^d r\, \Gamma(r) \sim \xi^{-p+d} \sim t^{\frac{p-d}{D_t}} \qquad d-2+\eta$$

$$\Rightarrow \eta = 2 - \gamma D_t$$

to find a relation for δ we rescale such that $h' = 1 = \left(\frac{\xi'}{\xi}\right)^{-D_h} h$

$$\Rightarrow \quad f(h,t) = h^{\frac{d}{D_h}} \underset{\substack{\uparrow \\ \text{regular at } h'=1}}{f\left(1, \left(\frac{\xi'}{\xi}\right)^{-\frac{D_t}{D_h}} t\right)}$$

$$\Rightarrow M = \partial_h f = h^{\frac{d}{D_h} - 1}$$

$$\Rightarrow \delta = \frac{d - D_h}{D_h}$$

all 6 critical exponents have been expressed in terms of D_t and D_h

$\Rightarrow$ $\exists$ only 2 independent critical exponents

<u>exercise</u>: show that these relations lead to the previously derived relations

5) Landau - Ginzberg theory

5a) Landau Functional

$$L = L(K_1, \ldots, K_n, \eta)$$

coupling constants   order parameter

$$L = \mathcal{L}\, V$$

State of system is determined by the absolute minimum of $\mathcal{L}$, i.e. $\frac{\partial \mathcal{L}}{\partial \eta} = 0$

LG theory is only valid near T_c because $\eta \to 0$ for $T \to T_c$ $\Rightarrow$ we can make an expansion into powers of η

requirements for $\mathcal{L}$

- $\mathcal{L}$ should have the symmetries of the system
- For inhomogeneous system with a space dependent order parameter, $\eta(r)$, $\mathcal{L}$ is a local function of $\eta(r)$, i.e., it depends only on a finite number of derivatives of $\eta(r)$
- For $T \to T_c$, $\mathcal{L}$ can be expanded into powers of η. For a continuous phase transition, η is small near T_c, and we can break off the expansion at the fourth term

$$\mathcal{L} = a_0 + a_1\eta + a_2\eta^2 + a_3\eta^3 + a_4\eta^4$$

The a_k are analytical functions of the coupling constants

- $T < T_c$ ordered phase
  $T > T_c$ disordered phase with $\eta = 0$

5b) $\mathcal{L}$ for the Ising universality class

Z_2 symmetry; invariance under $\eta \to -\eta$

$$\mathcal{L} = a_0 + a_1\eta + a_2\eta^2 + a_3\eta^3 + a_4\eta^4$$

$\eta \to -\eta$ $\quad -a_1\eta \quad\quad -a_3\eta^3$

$\Rightarrow \quad a_1 = a_3 = 0$

only minimum is relevant $\Rightarrow \quad a_0 = 0$

a_2 and a_4 are analytical functions of the temperature

$$a_2 = a_2^0 + a_2^1\, t$$
$$a_4 = a_4^0 + a_4^1\, t$$

- $a_4^0 > 0$, otherwise L does not have a real minimum
  $\Rightarrow$ $a_4^1\, t$ is irrelevant and we put $a_4^1 = 0$.