

Alternative relations

$$\left. \begin{aligned} \beta \delta &= \nu (d+2-\eta) / 2 \\ \beta &= \nu (d-2+\eta) / 2 \end{aligned} \right\} \Rightarrow \beta \delta - \beta = (2-\eta) \nu$$

$$\Rightarrow \beta (\delta - 1) = \gamma$$

Widom

$$2\beta + \nu = 2\nu - 2\nu + \nu\eta = -2\gamma$$

$$\Rightarrow \alpha + 2\beta + \gamma = 2 \quad \text{Rushbrook}$$

4c) Explanation

Near the critical point there is only one length scale, the correlation length ξ .

$$\xi = \xi(h, t) \quad h = \frac{H}{2T}$$

central idea: physics near the critical point is scale invariant, physics stays the same if we change the length scale

$$\xi^{-d} F(h', t') = \xi^d f(h, t)$$

↑ has dimension length^{-d}

$$h' = \left(\frac{\xi'}{\xi}\right)^{-b_h} h \quad t' = \left(\frac{\xi'}{\xi}\right)^{-D_t} t$$

$$\Rightarrow \xi'^{-d} f\left(\left(\frac{\xi'}{\xi}\right)^{-b_h} h, \left(\frac{\xi'}{\xi}\right)^{-D_t} t\right) = \xi^d f(h, t)$$

By changing the length scale we can always transform to $t'=1$

$$\Rightarrow \left(\frac{r'}{r}\right)^{-D_t} t = 1$$

$$\Rightarrow f(h, t) = t^{\frac{d}{D_t}} f\left(t^{-\frac{D_h}{D_t}} h, t=1\right)$$

at $t=1$ f is nonsingular

$$\epsilon \sim \partial_t^{\alpha} f(h, t=0) \sim t^{\frac{d}{D_t} - \alpha}$$

$$\Rightarrow \alpha = 2 - \frac{d}{D_t}$$

$$\frac{\Gamma}{V} = -\frac{\partial F}{\partial H} \Big|_{h=0} \sim t^{\frac{d}{D_t} - \frac{D_h}{D_t}} \Rightarrow \beta = \frac{d - D_t}{D_t}$$

$$\chi = -\frac{1}{V} \frac{\partial \Gamma}{\partial H} \sim \partial_h^{\gamma} F \Big|_{h=0} \sim t^{\frac{d}{D_t} - \frac{2D_h}{D_t}}$$

$$\Rightarrow \gamma = \frac{2D_h - d}{D_t}$$

$$f \sim \xi^{-d} \sim t^{\Delta d}$$

$$\Rightarrow \nu = \frac{1}{D_t}$$

$$\chi \sim \int d^d r \Gamma(r) \sim \xi^{-p+d} \sim t^{\frac{p-d}{D_t}}$$

$$\Rightarrow k = 2 - \gamma D_t$$

to find a relation for δ we rescale

$$\text{such that } \xi' = 1 = \left(\frac{\xi'}{\xi}\right)^{-D_h} h$$

$$\Rightarrow f(h, t) = h^{\frac{\lambda}{D_h}} f\left(1, \left(\frac{t}{h}\right)^{-\frac{D_t}{D_h}}\right)$$

↑
regular at $h=1$

$$\Rightarrow \gamma = \partial_h f = h^{\frac{\lambda}{D_h} - 1}$$

$$\Rightarrow \delta = \frac{d - D_h}{D_h}$$

all 6 critical exponents have been expressed in terms of D_t and D_h

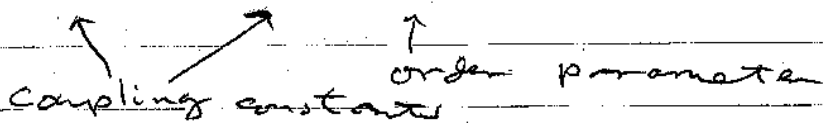
\Rightarrow \exists only 2 independent critical exponents

exercise: show that these relations lead to the previously derived relations

5) Landau - Ginzberg theory

5a) Landau Functional

$$L = L(k_1, \dots, k_n, \eta)$$



$$L = \int \mathcal{L} V$$

State of system is determined by the absolute minimum of \mathcal{L} , i.e. $\frac{\partial \mathcal{L}}{\partial \eta} = 0$

LG theory is only valid near T_c because $\eta \rightarrow 0$ for $T \rightarrow T_c \Rightarrow$ we can make an expansion into powers of η

Requirements for \mathcal{L}

- \mathcal{L} should have the symmetries of the system
- For inhomogeneous system with a space dependent order parameter, $\eta(r)$, \mathcal{L} is a local function of $\eta(r)$, i.e., it depends only on a finite number of derivatives of $\eta(r)$
- For $T \rightarrow T_c$, \mathcal{L} can be expanded into powers of η . For a continuous phase transition, η is small near T_c , and we can break off the expansion at the fourth term

$$\rho = a_0 + a_1 \eta + a_2 \eta^2 + a_3 \eta^3 + a_4 \eta^4$$

The a_k are analytical functions of the coupling constants

- $T < T_c$ ordered phase
- $T > T_c$ disordered phase with $\eta = 0$

5b) ρ for the Ising universality class

\mathbb{Z}_2 symmetry; invariance under $\eta \rightarrow -\eta$

$$\rho = a_0 + a_1 \eta + a_2 \eta^2 + a_3 \eta^3 + a_4 \eta^4$$

$\eta \rightarrow -\eta$ $-a_1 \eta$ $-a_3 \eta^3$

$$\Rightarrow a_1 = a_3 = 0$$

only minimum is relevant $\Rightarrow a_0 = 0$

a_2 and a_4 are analytical functions of the temperature

$$a_2 = a_2^0 + a_2^1 t$$

$$a_4 = a_4^0 + a_4^1 t$$

- $a_4^0 > 0$, otherwise L does not have a real minimum

$\Rightarrow a_4^1 t$ is irrelevant and we put $a_4^1 = 0$.