Spectral Density of the SYK model

$$Z(\beta) = \int dE\, \rho(E)\, e^{-\beta E}$$

$$\Rightarrow \quad \rho(E) = \underbrace{\int_{c-i\infty}^{c+i\infty} e^{\beta E}\, Z(\beta)\, d\beta}_{\text{inverse Laplace transform}}$$

$$Z(\beta) = e^{\beta E_0} + \underbrace{\frac{N}{2}\log 2 - \frac{N\pi^2}{4q^2}}_{\text{entropy}} + \underbrace{\frac{J}{q^2\beta}}_{\text{low T expansion}}$$

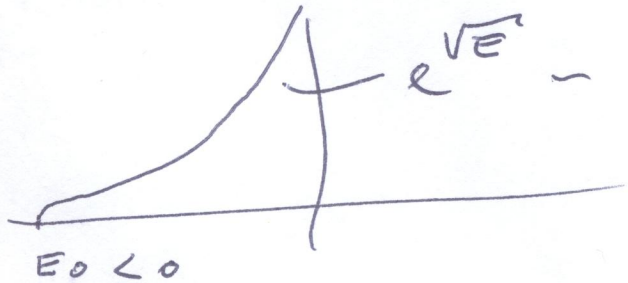
Guess for low energy part of spectral density based on integral

$$\int_0^\infty \sinh\sqrt{E}\, e^{-\beta E}\, dE = \qquad = \frac{e^{\frac{1}{4\beta}}\sqrt{\pi}}{2\beta^{3/2}}$$

$$\Rightarrow \int_0^\infty \sinh\sqrt{\tfrac{1}{4}(E+E_0)}\, e^{-\beta E}\, dE = 4e^{\beta E_0}\, \frac{e^{\frac{1}{4\beta}}\sqrt{\pi}}{2\beta^{3/2}}$$

$$\Rightarrow \rho(E) = e^{\frac{N}{2}\log 2 - \frac{N\pi^2}{4q^2} + \beta E_0} \times \frac{\beta^{3/2}}{2\sqrt{\pi}}$$

$$\times \sinh\sqrt{(E+\underset{\substack{\wedge \\ 0}}{E_0})/4}$$



Calculation of the Spectral density

$$\int \rho(E)\, e^{iEt}\, dt = \int \rho(E) \sum_k \frac{(iEt)^k}{k!}\, dE$$

$$= \sum_k \frac{(it)^k}{k!} M_k$$

moments

$$M_k = \int \rho(E)\, E^k\, dE$$

So we can calculate the spectral density if we know the moments.

$$H = \sum_\alpha J_\alpha \Gamma_\alpha$$ ← product of q γ matrices $\Gamma_\alpha = \gamma_{\alpha_1} \cdots \gamma_{\alpha_q}$

$\binom{N}{q}$ different indices

$\Gamma_\alpha^2 = 1$ in normalization where $\gamma_\alpha^2 = 1$

J_α is Gaussian distributed

So the average is given by the sum over all pairwise contractions.

$$\overline{\frac{1}{D} \mathrm{Tr}\, H^2} = \frac{1}{D} \sum_{\alpha\beta} \overline{\mathrm{Tr}\, J_\alpha \Gamma_\alpha J_\beta \Gamma_\beta} = \frac{1}{D} \sum_\alpha \overline{J_\alpha J_\alpha}\, \mathrm{Tr}\, \Gamma_\alpha^2$$

$$= \binom{N}{q} \sigma^2$$

$$\frac{1}{D} \overline{\mathrm{Tr}\, H^4} = \frac{1}{D} \sum_{\alpha\beta\gamma\delta} \overline{\mathrm{Tr}\, J_\alpha J_\beta J_\gamma J_\delta \Gamma_\alpha \Gamma_\beta \Gamma_\gamma \Gamma_\delta}$$

pairwise contractions

$$\overline{\gamma_\alpha \gamma_\beta \gamma_\gamma \gamma_\delta} = \gamma_\alpha \gamma_\alpha \gamma_\beta \gamma_\beta + \gamma_\alpha \gamma_\beta \gamma_\beta \gamma_\alpha + \gamma_\alpha \gamma_\beta \gamma_\alpha \gamma_\beta$$

each contraction gives σ^2

$$\frac{1}{D} \mathrm{Tr}\, H^4 = \sum_{\alpha\beta} \frac{\sigma^4}{D} \mathrm{Tr}\, \Gamma_\alpha^2 \Gamma_\beta^2 + \frac{\sigma^4}{D} \mathrm{Tr}\, \Gamma_\alpha \Gamma_\beta \Gamma_\beta \Gamma_\alpha$$

$$+ \frac{\sigma^4}{D} \mathrm{Tr}\, \underbrace{\Gamma_\alpha \Gamma_\beta \Gamma_\alpha \Gamma_\beta}_{\text{we have to commute } \Gamma_\alpha \Gamma_\beta}$$

$=$

α, β have no indices in common

$$\Gamma_\alpha \Gamma_\beta = \Gamma_\beta \Gamma_\alpha$$

α, β have one index in common

$$\gamma_\alpha \gamma_{\beta_1} \gamma_{\gamma_1} \gamma_{\delta_1} \gamma_\alpha \gamma_{\beta_2} \gamma_{\gamma_2} \gamma_{\delta_2} = - \gamma_\alpha \gamma_{\beta_2} \gamma_{\gamma_2} \gamma_{\delta_2} \gamma_\alpha \gamma_{\beta_1} \gamma_{\gamma_1} \gamma_{\delta_2}$$

$$\Gamma_\alpha \Gamma_\beta = (-1)^p \Gamma_\beta \Gamma_\alpha$$

p is the number of common indices.

$$\Rightarrow \frac{1}{D} \mathrm{Tr}\, \Gamma_\alpha \Gamma_\beta \Gamma_\alpha \Gamma_\beta = \binom{N}{q} \sum_{j=0}^{q} (-1)^j \binom{q}{j} \binom{N-q}{q-j}$$

$$M_4 = \binom{N}{q}^2 \sigma^4 \Big(2 + \underbrace{\binom{N}{q}^{-1} \sum_{j=1}^{q} (-1)^j \binom{q}{j}\binom{N-q}{q-j}}_{\equiv \eta} \Big)$$

Let us first look at the limit that $q \ll N$. Then almost all of the Γ_α and Γ_β commute. So we can treat them as independent.

$$\overline{\frac{1}{D} \mathrm{Tr}\, H^{2p}} = \underbrace{\sum \overline{J_{\alpha_1} J_{\alpha_2} \cdots J_{\alpha_{2p}}}}_{(2p-1)!!\, \sigma^{2p} \binom{N}{q}^p} \underbrace{\frac{1}{D} \mathrm{Tr}\, \Gamma_{\alpha_1} \Gamma_{\alpha_2} \cdots \Gamma_{\alpha_{2p}}}_{=1}$$

These are the moments of a Gaussian distribution.

The next approximation is to treat all crossing of contractions as independent

$$\gamma_\alpha \gamma_\beta \gamma_\gamma \gamma_\alpha \gamma_\beta \gamma_\gamma \quad \mathrm{Tr}\, \Gamma_\alpha \Gamma_\beta \Gamma_\gamma \Gamma_\alpha \Gamma_\beta \Gamma_\gamma$$

three crossing.

so result is given by $\sigma^6 \binom{N}{q}^3 \eta^3$

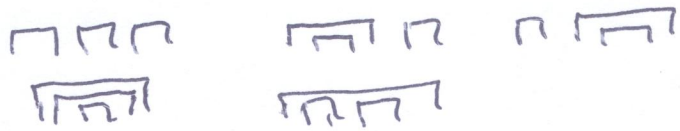
So $M_{2p} = M_2^p \sum c_K \eta^K$

number of diagrams with K crossing

$p=2$ 2 diagrams with zero crossing
1 diagram with one crossing

$p=3$ $(2p-1)!! = 15$ diagrams

zero crossings



one crossing

two crossings

three crossing

The c_k are given by the Riordan–Touchard formula

$$\sum c_k q^k = \frac{1}{(1-q)^p} \sum_{k=-p}^{p} (-1)^k q^{k(k-1)/2} \binom{2p}{p+k}$$

For $p=3$ $\quad 5 + 6q + 3q^2 + q^3$

The corresponding spectral density is also known

$$\rho^{QH}(E) = \sqrt{1-\frac{E^2}{E_0^2}} \prod_{k=1}^{\infty} \left(1 - \frac{4E^2}{E_0^2} \frac{1}{2+q^k+q^{-k}}\right)$$

$$E_0^2 = \frac{4\sigma^2}{1-q}$$

This is the density function for the Q-Hermite polynomials

$$H_{n+1}^{n}(x) = x H_n^n(x) - \sum_{k=0}^{n-1} q^k H_{n-1}^{n}(x)$$