Mean field equations are translational invariant. Just substitute $\tau \to \tau + a$
$\tau' \to \tau' + a$

Green's function

$$G(t) = \langle \chi(t)\chi(0)\rangle = \langle \chi(0)\,\chi(-t)\rangle = -\langle \chi(-t)\,\chi(0)\rangle$$
$$= -G(-t)$$

The mean field equations have a diffeomorphisme invariance

$$\tau \to f(\tau)$$

If $G(\tau, \tau')$ is a solution, then also

$\left(f'(\tau)\; f'(\tau')^{\Delta}\right) G(f(\tau), f(\tau'))$ is also a solution

$$\Sigma(\tau, \tau') = J^2\, G^{q-1}(\tau, \tau')$$

$$(\Sigma . G)' = \int G(f(\tau), f(s))\, \Sigma(f(s), f(\tau''))\, f'(\tau)^{\Delta} f'(s)^{4\Delta} (f'(\tau''))^{3\Delta}\; \cancel{df(s)}\; ds$$

$$4\Delta = 1 \qquad f'(s)^{4\Delta}\, ds = df(s)$$

integrate over $f(s)$

$$= -\delta(f(\tau), f(\tau''))\, f'(\tau)^{\Delta}\, f'(\tau'')^{3\Delta}$$

$$= -\frac{\delta(\tau - \tau'')}{|f'(\tau)|}\, |f'(\tau)|^{4\Delta} = -\delta(\tau - \tau')$$

$f(\tau)$ should be an increasing function

This can be used to construct a solution at finite temperature

$$f(\tau) = \tan \frac{\pi\tau}{\beta}$$


$$f'(\tau) = \frac{\pi}{\beta} \cos^{-2} \frac{\tau\pi}{\beta}$$

$$\text{So } G(\tau) = b \sin\tau \left(\frac{\beta}{\pi}\right)^{-2\Delta} \cos^{-2\Delta}\left(\frac{\pi\tau}{\beta}\right) \frac{1}{\tan^{2\Delta} \frac{\tau\beta}{\pi}}$$

put $\tau' = 0$

Not all diffeomorphisms give a new solution. This is the case if

$$\langle f'(\tau) f'(0))^{\Delta} \frac{1}{(f(\tau)-f(0))^{2\Delta}} = \frac{1}{\tau^{2\Delta}}$$

$$\Rightarrow \frac{f'(\tau) f(0)}{(f(\tau)-f(0))^2} = \frac{1}{\tau^2}$$

$$\Rightarrow \frac{d}{d\tau}\left( \frac{1}{f(\tau)-f(0)} f'(0) = \frac{d}{d\tau}\frac{1}{\tau}$$

$$\Rightarrow \frac{f'(0)}{f(\tau)-f(0)} = c + \frac{1}{\tau}$$

$$\Rightarrow f(\tau) = f(0) + \frac{c}{f(0)} + \frac{1}{f(0)\tau}$$

translations are also a new symmetry

$\Rightarrow$ full invariance group

$$f(\tau) = f(0) + \frac{c}{f'(0)} + \frac{1}{f'(0)(\tau+a)}$$

This is $Sl(2,R)$.

$\Rightarrow$ Goldstone manifold of solutions $\quad Diff(1,R)/Sl(2,R)$

<u>Large q expansion</u>

In the action q occurs as

$$\frac{1}{q} G^{q-1} \Sigma$$

For $q \to \infty$ this interaction term is suppressed and we can expand G about the free propagator

$$\frac{d}{d\tau} G_{free}(\tau) = -\delta(\tau)$$

$$\Rightarrow \quad G_{free} = \frac{1}{2} \operatorname{sign}(\tau)$$

$$G = G_{free} + \operatorname{sign}\tau \, g(\tau)/q$$

$$\frac{1}{q} G^q = \frac{1}{q}\left(\frac{1}{2}\operatorname{sign}\tau + \operatorname{sign}\tau\,\frac{g(\tau)}{q}\right)^q$$

$$\Sigma = J^2 G^{q-1} = J^2 2^{1-q} \operatorname{sign}(\tau) \left(1 + \frac{g}{q}\right)^{q-1}$$
$$= J^2 2^{1-q} \operatorname{sign}(\tau)\, e^{g} \quad q \to \infty$$

$$G(\omega) = \underbrace{-\frac{1}{i\omega}}_{\text{Fourier transform of } \frac{1}{2}\operatorname{sign}(\tau)} + \frac{1}{2q} \underbrace{\left([\operatorname{sign} g](\omega)\right)}_{\text{Fourier transfom of } \operatorname{sign}(\tau) g(\tau)}$$

$$\Rightarrow \frac{1}{G(\omega)} = -i\omega + \frac{\omega^2}{q} [\operatorname{sign} g](\omega)$$
$$\equiv -i\omega - \Sigma(\omega)$$

Fourier transform back

$$-\partial_\tau^2 \left(\operatorname{sign}(\tau)\, g(\tau)\right) = -2q J^2 2^{-q} \operatorname{sign}(\tau) e^{g(\tau)}$$

This is a differential equation for $g(\tau)$ which can be solved

$$e^{g(\tau)} = \frac{c^2}{J^2} \frac{1}{\sin^2(c|\tau| + \tau_0)}$$

At very short time we should find the free fermion result

$\Rightarrow \quad g(0) = 0 \qquad g(\beta) = g_0$

periodic boundary conditions

The solution that satisfies these boundary conditions is given by

$$e^{g(\tau)} = \frac{\cos^2 \pi v}{\cos^2\left(\frac{\pi v}{2} - \frac{|\tau|}{\beta}\right)} \qquad \cos\frac{\pi v}{2} = \frac{c^2}{\mathcal{J}^2}$$

$$c = \frac{\pi\beta}{\beta^2}$$

## Action of large q solution

we have that $\mathcal{J}\partial_{\mathcal{J}} \log Z = \beta\partial_\beta \log Z$

because the partition function only depends on the combination $\beta\mathcal{J}$.

$$\mathcal{J}\partial_{\mathcal{J}} \log Z = \mathcal{J}\partial_{\mathcal{J}}(-\beta F)$$

only $\mathcal{J}$ dependent part of the action contributes

$$\mathcal{J}^2 = \frac{q J^2}{2^{\frac{q-1}{2}}} \qquad \text{action} \sim \frac{J}{q} G^q \sim \frac{\mathcal{J}^2}{q^2} G^q$$

$$= \cancel{\mathcal{J}\partial_{\mathcal{J}}} \quad \frac{\beta\mathcal{J}^2}{2q}\int_0^\beta d\tau\, e^{g(\tau)}$$

$$\sigma^2 = \frac{J^2 (q-1)!}{N^{q-1}}$$

↑ variance of $J_{\alpha\beta\gamma\delta}$

$$\frac{N^4\sigma^2}{q!} G^q = \frac{N^4 J^2 (q-1)!}{N^3 q!} G^q = \frac{N J^2}{q} G^q$$

$$\Sigma G = J^2 G^q$$

$$= \frac{\beta J^2}{2q} \frac{\beta}{\pi v} \cos^2 \frac{\pi v}{2} \int_0^\beta d\tau \frac{d}{d\tau} \tan \pi v \left( \frac{1}{2} - \frac{|\tau|}{\beta} \right)$$

$$= \frac{\beta J^2}{q} \frac{\beta}{\pi v} \cos^2 \frac{\pi v}{2} \tan \frac{\pi v}{2}$$

relation between $v$ and $\beta J$

$$\frac{d\beta J}{\beta J} = \frac{dv}{v} + \frac{\pi\, dv}{2} \tan \frac{\pi v}{2}$$

This gives $$\frac{v}{1 + \frac{\pi v}{2} \tan \frac{\pi v}{2}} \partial_v (-\beta F) = N \frac{\pi v}{q} \tan \frac{\pi v}{2}$$

$v \to 0$ should give zero coupling limit.

$$-\beta F \big|_{J=0} = \frac{N}{2} \log 2$$

low temperature expansion

$\beta \to \infty$ $\quad v \to 1 + \delta v$

$$\Rightarrow \pi \frac{\delta v}{2} = \frac{\pi}{\beta J}$$

$$-\frac{\beta F}{N} = \frac{1}{2} \log 2 - \frac{\pi^2}{4q^2} + \frac{\beta J}{q^2}$$

$\Rightarrow$ the zero temperature entropy

is extensive

$$= \frac{\beta \mathcal{J}^2}{2q}\,\frac{\beta}{\pi v}\cos^2\frac{\pi v}{2}\int_0^\beta d\tau\,\frac{d}{d\tau}\tan \pi v\left(\frac{1}{2}-\frac{|\tau|}{\beta}\right)$$

$$= \frac{\beta \mathcal{J}^2}{q}\,\frac{\beta}{\pi v}\cos^2\frac{\pi v}{2}\tan\frac{\pi v}{2}$$

relation between $v$ and $\beta\mathcal{J}$

$$\frac{d\beta\mathcal{J}}{\beta\mathcal{J}} = \frac{dv}{v} + \frac{\pi\, dv}{2}\tan\frac{\pi v}{2}$$

This gives $\dfrac{v}{1+\frac{\pi v}{2}\tan\frac{\pi v}{2}}\,\partial_v(-\beta F) = N\dfrac{\pi v}{q}\tan\dfrac{\pi v}{2}$

$v \to 0$ should give zero coupling limit.

$$-\beta F\big|_{\mathcal{J}=0} = \frac{N}{2}\log 2$$

low temperature expansion
$\beta\to\infty$ $\quad v \to 1+\delta v$

$$\Rightarrow \pi\frac{\delta v}{2} = \frac{\pi}{\beta\mathcal{J}}$$

$$-\frac{\beta F}{N} = \frac{1}{2}\log 2 - \frac{\pi^2}{4q^2} + \frac{\beta\mathcal{J}}{q^2}$$

$\Rightarrow$ the zero temperature entropy