So

$$\log Z = \beta E_0 + \int_{-1}^{1} \rho(E) dE (1 + e^{-2\beta E}) \tag{261}$$

with E_0 the sum of the single particle energies and the single particle level density

$$\rho(E) = \frac{N}{\pi}\sqrt{1 - E^2}. \tag{262}$$

We thus have that

$$\log Z = \beta E_0 + \frac{N}{\pi}\int_0^{\pi} d\theta \cos^2\theta \log(1 + e^{-2\beta \sin\theta}). \tag{263}$$

The low temperature limit is obtained by expanding the logarithm

$$\log Z = \beta E_0 + N\frac{\pi}{12\beta}. \tag{264}$$

So the zero temperature entropy, which would be the constant term in $\log Z$ vanishes. The hightemperature limit of the partition function is given by

$$\log Z = \beta E_0 + \frac{N}{2}\log 2 + \cdots, \tag{265}$$

which is the logarithm of the total number of states.

15.5 Calculation of the Spectral Density

The most straigtforward way to calculate the spectral density is to use the moments to obtain the Fourier transform of the spectral density,

$$\rho(t) = \int dE \rho(E) e^{iEt} = \sum \frac{(Et)^{2k}(-1)^k}{k!} M_{2k}, \tag{266}$$

where we have assumed that $\rho(E)$ is an even function of E. In the limit that $N \gg q$, the indices of the terms contributing to the Hamiltonian are almost always different, and we can assume that the Γ_α commute. Summing over all Wick contractions we obtain

$$M_{2p} = (2p-1)!! M_2^p. \tag{267}$$

These are the moments of a Gaussian distribution.

15.6 Path Integral Formulation of the SYK Model

For Dirac fermions we know very well how to write down the path integral of fermion fields. Just replace the fermion operators by complex Grassmann variables. For Majorano fermions we replace the real fermion operators, which can be written as $\chi = c + c^\dagger$, by real Grassmann variables. The kinetic term of the Lagrangian is given by

$$L_0 = \int d\tau \chi(\tau) \frac{d}{d\tau}\chi(\tau). \tag{268}$$

We thus obtain the path integral

$$Z = \int D\chi(\tau) e^{-\int d\tau \chi \frac{d}{d\tau}\chi - \sum_{\alpha\beta\gamma\delta} J_{\alpha\beta\gamma\delta}\chi_\alpha\chi_\beta\chi_\gamma\chi_\delta}. \tag{269}$$

The free energy can be obtained using the replica trick

$$\log Z = \lim_{n\to 0} \frac{Z^n - 1}{n}. \tag{270}$$

To calculate Z^n we given the fields an extra index, the replica index. However, since thermodynamic properties do no depend on the number of replicas, we will do the calculation for one replica.

The integral over the Gaussian random variables can be done by completing square. Denoting the variance by σ^2 and using

$$\int dJ e^{-J^2/2\sigma^2 + JA} \sim e^{\frac{1}{2}\sigma^2 A^2} \tag{271}$$

we obtain

$$Z = \int D\chi(\tau) e^{-\int d\tau \chi \frac{d}{d\tau}\chi + \frac{1}{2}\sigma^2 \sum_{\alpha\beta\gamma\delta}\int d\tau d\tau' \chi_\alpha(\tau)\chi_\beta(\tau)\chi_\gamma(\tau)\chi_\delta(\tau)\chi_\alpha(\tau')\chi_\beta(\tau')\chi_\gamma(\tau')\chi_\delta(\tau')} \tag{272}$$

Next we introduce new variables by inserting a δ function

$$\delta(G + \frac{1}{N}\sum_\alpha \chi(\tau)\chi(tau')) = \int D\Sigma(\tau) e^{\overset{N}{\frac{1}{2}}\int d\tau d\tau' \Sigma(\tau,\tau')(G(\tau,\tau') + \frac{1}{N}\sum_\alpha \chi(\tau)\chi(\tau'))}. \tag{273}$$

The integral over $\Sigma(\tau)$ has to be over the imaginary axis, but we continue it to real axis. We thus find the partition function (we wrote it down for arbitrary q)

$$Z = \int D\chi(\tau) e^{-\int_0^\beta d\tau \chi \frac{d}{d\tau}\chi + \overset{N^4}{\frac{\sigma^2}{q!}}\int_0^\beta d\tau d\tau' G(\tau,\tau')q + \overset{N}{\frac{1}{2}}\int d\tau d\tau' \Sigma(\tau,\tau')(G(\tau',\tau) + \frac{1}{N}\sum_\alpha \chi(\tau)\chi(tau')}} \tag{274}$$

Now the the integral of $\chi(\tau)$ can be done. It gives a Pfaffian. The action is thus given by

$$S = -\int\int d\tau d\tau' \left[\frac{N}{2}\mathrm{Tr}\log(\overset{d/d\tau}{\delta(\tau,\tau')} + \Sigma(\tau,\tau')) + \frac{N^{\overset{q}{4}}\sigma^2}{q!}G^q(\tau,\tau') + N\Sigma(\tau,\tau')G(\tau',\tau)\right]. \tag{275}$$

Because of our choice of the variance, N only appears as a prefactor. For large N we can evaluate the integral by a saddle-point approximation.

$$\begin{aligned} \int d\tau''(\delta(\tau,\tau'')\frac{d}{d\tau} + \Sigma(\tau,\tau''))G(\tau'',\tau') &= -\delta(\tau,\tau', \\ \Sigma(\tau,\tau') = J^2 G^{q-1}(\tau,\tau') & \end{aligned} \tag{276}$$

The second equation is valid point by point. Because the source term is translationally invariant, the solutions of the Dyson-Schwinger equations are also translationally invariant. It is simplest to solve these equation by Fourier transforming the first equation

$$\begin{aligned} \Sigma(t-s) &= \frac{1}{\sqrt{2\pi}}\int dt e^{iE(t-s)}\Sigma(E), \\ G(s-t) &= \frac{1}{\sqrt{2\pi}}\int dE' e^{iE(s-t')}G(E'). \end{aligned} \tag{277}$$

After integration over s and then integrating $\delta(E-E')$, the first equation becomes

$$\int dE\Sigma(E)G(E)e^{iE(t-t')} = -\frac{1}{2\pi}\int dE e^{iE(t-t')}. \tag{278}$$

We thus find that

$$\Sigma(E)G(E) = -\frac{1}{2\pi}. \tag{279}$$

We make the Ansatz

$$G(E) = -ib\,\mathrm{sign}(E)/\sqrt{|E|}^{\alpha}. \tag{280}$$

Then

$$\Sigma(E) = -i\frac{1}{2\pi b}\mathrm{sign}(E)\sqrt{|E|}^{\pm\alpha} \tag{281}$$

The Fourier transform of $\mathrm{sign}(E)\sqrt{|E|}^{\alpha}$ is given by

$$\begin{aligned}
\frac{1}{\sqrt{\pi}}\int dE e^{-itE}\mathrm{sign}(E)\sqrt{|E|}^{\alpha} &= \frac{-i}{\sqrt{\pi}}\int dE\sin(tE)\mathrm{sign}(E)\sqrt{|E|}^{\alpha} \\
&= \frac{-2i}{\sqrt{\pi}}\int_0^\infty dE\sin(Et)\mathrm{sign}(t)E^{\alpha/2} \\
&= \frac{-2i}{\sqrt{\pi}}t^{-\frac{\alpha}{2}-1}\mathrm{sign}(t)\int_0^\infty dE\sin(E)E^{\alpha/2}\cancel{E}^{\alpha/2} \\
&= \frac{-2i}{\sqrt{\pi}}t^{-\frac{\alpha}{2}-1}\mathrm{sign}(t)\cos(\alpha\pi/4)\Gamma(1+\alpha/2)
\end{aligned} \tag{282}$$

Inserting this in the second saddle point equation, we obtain

$$\begin{aligned}
&\left[-i\frac{1}{2\pi b}(\frac{(-2i)}{\sqrt{\pi}}t^{\alpha/2-1}\cos(-\alpha\pi/4)\Gamma(1-\alpha/2)\right]^{(\overset{q-1}{\cancel{q-1}})}\mathrm{sign}(t) \\
= &\quad -ib\frac{-2i}{\sqrt{\pi}}t^{-\frac{\alpha}{2}-1}\mathrm{sign}(t)\cos(\alpha\pi/4)\Gamma(1+\alpha/2)
\end{aligned} \tag{283}$$

This gives that $\alpha = 2(1-2\Delta)$ so that \Delta = 1/q

$$G(t) = b\mathrm{sign}(t)/|t|^{2\Delta}. \tag{284}$$

(q-1)(\al/2-1) = -\al/2-1

.

The coefficient b is also determined by this equation. q\al/2=q-2

15.7 Diffeomorphism Invariance

The mean field equations are inhomogenous equations, and will have a unique solution subject to boundary conditions. We see immediately that if $G(t,t')$ is a solution than also $G(t+a,t'+a)$ is a solution. So for translational invariant boundary conditions, we have that the solution only depends on the difference of its arguments, ile $G(t,t') = G(t-t')$. This also implies

$$G(t) = \langle\chi(t)\chi(0)\rangle = \langle\chi(0)\chi(-t)\rangle = -\langle\chi(-t)\chi(0)\rangle = -G(-t). \tag{285}$$

Path integral formulation of the SYK model

Dirac fermions: replace fermion operators by complex Grassmann variables

Majorana fermions; $\chi = c + c^+$

so the recipe is to replace χ by real Grassmann variables

Kinetic term $\quad L = \int_0^{\beta} d\tau \; \chi(\tau) \frac{d}{d\tau} \chi(\tau)$ ← inverse temperature

So the path integral corresponding to the SYK Hamiltonian is given by

$$Z = \int D\chi(\tau) \; e^{-\int_0^{\beta} \chi_\alpha \frac{d}{d\tau} \chi_\alpha - \sum_{\alpha\beta\gamma\delta} J_{\alpha\beta\gamma\delta} \chi_\alpha \chi_\beta \chi_\gamma \chi_\delta}$$

↑ sum over α

The free energy is given by $\log Z$ which can be evaluated by means of the replica trick

$$\log Z = \lim_{n \to 0} \frac{Z^n - 1}{n}$$

For Z^n we can evaluate the disorder average where the fields just get an additional replica index

We will study the mean field limit of the partition function, and in this limit the results do not depend on the replica index. We can just evaluate the partition function for $n=1$.

The integral over $J_{\alpha\beta\gamma\delta}$ is Gaussian and can be evaluated by completing squares

$$\int dJ\, e^{-\frac{J^2}{2\sigma^2} + JA} \sim e^{\frac{1}{2}\sigma^2 A^2}$$

This gives

$$Z = \int DX(\tau)\, e^{-\int d\tau\, X \frac{d}{d\tau} X + \frac{1}{2}\sigma^2 \int d\tau\, d\tau'\, X_\alpha(\tau) X_\beta(\tau) X_\gamma(\tau) X_\delta(\tau)\, X_\alpha(\tau') X_\beta(\tau') \times X_\gamma(\tau') X_\delta(\tau')}$$

Next we bosonize by inserting a δ-function

$$\delta\Big(G + \frac{1}{N} \sum_\alpha X_\alpha(\tau) X_\alpha(\tau')\Big) = \int d\Sigma(\tau)\, e^{\frac{1}{2}\int d\tau\, d\tau'\, \Sigma(\tau,\tau')\left(G(\tau,\tau') + \frac{1}{N}\sum_\alpha X_\alpha(\tau) X_\alpha(\tau')\right)}$$

Note that the integral over $\Sigma(\tau,\tau')$ has been continued from the imaginary axis to the real axis.

$$Z = \int dX(\tau)\, e^{-\int_0^\beta d\tau\, X(\tau) \frac{d}{d\tau} X(\tau) + \frac{\sigma^2}{q!} \int d\tau\, d\tau_1\, G^q(\tau,\tau') N + \frac{1}{2} \int d\tau\, d\tau'\, \Sigma(\tau,\tau')\left(G(\tau,\tau') + \frac{1}{N}\sum_\alpha X_\alpha(\tau) X_\alpha(\tau')\right)}$$

We can do the integral over $\chi(\tau)$.

This gives a Pfaffian (square root of a determinant)

We thus find

$$Z = \int d\cancel{\ }G\, d\Sigma\, e^{-\int S(G,\Sigma)\, d\tau d\tau'}$$

$$S(G,\Sigma) = -\int d\tau d\tau' \left[\frac{N}{2} \log\left(\delta(\tau,\tau') \frac{d}{d\tau} + \Sigma(\tau,\tau')\right) + \frac{N^q \sigma^2}{q!} G^q(\tau,\tau') + N\, \Sigma(\tau,\tau')\, G(\tau,\tau) \right]$$

The action $\sim N \Rightarrow$ For large N we can evaluate the partition function using Mean field theory.

The saddlepoint equations are given by

$$\int d\tau'' \left(\delta(\tau,\tau'') \frac{d}{d\tau} + \Sigma(\tau,\tau'') G(\tau'',\tau') \right) = -\delta(\tau,\tau')$$

$$\Sigma(\tau,\tau') = J^2 G^{q-1}(\tau,\tau')$$

Note that we only get a stable large N limit when $N^{q-1}\sigma^2$ is kept fixed for $N \to \infty$. This is exactly the reason why we chose $\sigma^2 \sim \frac{1}{N^{q-1}}$ in the definition of the model

The first saddle-point equation is a convolution. So we Fourier transform

$$\Sigma(t-s) = \frac{1}{\sqrt{2\pi}} \int dE\, e^{iE(t-s)} \Sigma(E)$$

$$G(s-t) = \frac{1}{\sqrt{2\pi}} \int dE\, e^{iE(s-t)} G(E)$$

We also assumed translational invariance.

At strong coupling we can ignore the $\frac{d}{d\tau}$ term.

After integrating over s and E we find

$$\int dE\, \Sigma(E) G(E) e^{iE(t-t')} = -\frac{1}{2\pi} \underbrace{\int dE\, e^{iE(t-t')}}$$

write δ-function as an integral

This results in $\Sigma(E) G(E) = -\frac{1}{2\pi}$

Ansatz $G(E) = -ib\, \text{sign}(E) \frac{1}{\sqrt{|E|^\alpha}}$

Then $\Sigma(E) = \frac{-i}{2\pi b} \text{sign}(E) \sqrt{|E|^\alpha}$

The second equation is simple in the time variables. So we Fourier transform back.

$$\frac{1}{\sqrt{\pi}} \int dE\, e^{-itE} \operatorname{sign}(E) |E|^{\alpha/2} =$$

$$= \frac{-i}{\sqrt{\pi}} \int dE \, \sin(tE) \operatorname{sign}(E) |E|^{\alpha/2}$$

$$= -\frac{2i}{\sqrt{\pi}} \int_0^\infty dE \, \sin(tE) \operatorname{sign}(E) |E|^{\alpha/2}$$

$$= \frac{-2i}{\sqrt{\pi}} t^{-\frac{\alpha}{2}-1} \operatorname{sign}(t) \int_0^\infty dE \sin(E) E^{\alpha/2}$$

$$= \frac{-2i}{\sqrt{\pi}} t^{-\frac{\alpha}{2}-1} \operatorname{sign}(t) \cos\frac{\alpha\pi}{4} \Gamma(1+\tfrac{\alpha}{2})$$

We can do the same for $\alpha \to -\alpha$ which gives the Fourier transform of $G(E)$.

Inserting this in the second saddle point equation we obtain

$$\left[\frac{-i}{2\pi b}\left(\frac{(2-i)}{\sqrt{\pi}}\right) t^{\frac{\alpha}{2}-1} \cos(-\tfrac{\alpha\pi}{4}) \Gamma(1-\tfrac{\alpha}{2})\right)^{q-1} \operatorname{sign}(t)$$

$$= -ib \frac{-2i}{\sqrt{\pi}} t^{-\frac{\alpha}{2}-1} \operatorname{sign}(t) \cos\frac{\alpha\pi}{4} \Gamma(1+\tfrac{\alpha}{2})$$

$$q \text{ is even} \Rightarrow \operatorname{sign}^{q-1}(t) = \operatorname{sign}(t)$$

Equating the time dependence on both sides we obtain

$$(\tfrac{\alpha}{2}-1)(q-1) = -\tfrac{\alpha}{2}-1 \Rightarrow \tfrac{\alpha}{2}(q-1+1) = q-1-1$$

$$\Rightarrow \alpha = 2\frac{(q-2)}{q} = 2(1-\tfrac{2}{q})$$

$$\Rightarrow G(t) \sim t^{+\frac{\alpha}{2}-1} \operatorname{sign}(t) \sim t^{-\frac{2}{q}} \ldots$$

$$\frac{1}{\sqrt{\pi}} \int dE\, e^{-itE} \operatorname{sign}(E) |E|^{\alpha/2} =$$

$$= \frac{-i}{\sqrt{\pi}} \int dE \, \sin(tE) \operatorname{sign}(E) |E|^{\alpha/2}$$

$$= -\frac{2i}{\sqrt{\pi}} \int_0^\infty dE \, \sin(tE) \operatorname{sign}(E) |E|^{\alpha/2}$$

$$= \frac{-2i}{\sqrt{\pi}} t^{-\frac{\alpha}{2}-1} \operatorname{sign}(t) \int_0^\infty dE \sin(E) E^{\alpha/2}$$

$$= \frac{-2i}{\sqrt{\pi}} t^{-\frac{\alpha}{2}-1} \operatorname{sign}(t) \cos\frac{\alpha\pi}{4} \Gamma(1+\frac{\alpha}{2})$$

We can do the same for $\alpha \to -\alpha$ which gives the Fourier transform of $G(E)$.

Inserting this in the second saddle point equation we obtain

$$\left[\frac{-i}{2\pi b}\left(\frac{(2-i)}{\sqrt{\pi}}\right) t^{\frac{\alpha}{2}-1} \cos(-\frac{\alpha\pi}{4}) \Gamma(1-\frac{\alpha}{2})\right)^{q-1} \operatorname{sign}(t)$$

$$= -ib \frac{-2i}{\sqrt{\pi}} t^{-\frac{\alpha}{2}-1} \operatorname{sign}(t) \cos\frac{\alpha\pi}{4} \Gamma(1+\frac{\alpha}{2})$$

q is even $\Rightarrow \operatorname{sign}^{q-1}(t) = \operatorname{sign}(t)$

Equating the time dependence on both sides we obtain

$$(\frac{\alpha}{2}-1)(q-1) = -\frac{\alpha}{2}-1 \Rightarrow \frac{\alpha}{2}(q^{-1}+1) = q-1-1$$

$$\Rightarrow \alpha = 2\frac{(q-2)}{q} = 2(1-\frac{2}{q})$$

$$\Rightarrow G(t) \sim t^{+\frac{\alpha}{2}-1} \operatorname{sign}(t) = \ldots^{-\frac{2}{q}} \ldots$$