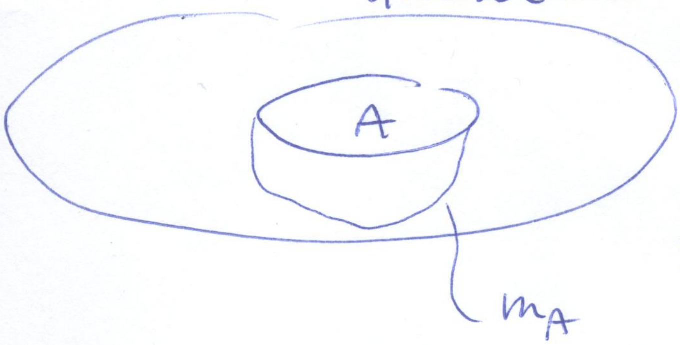
Ryu - Takayanagi formula

conformal field theory $\Leftrightarrow$ Anti de Sitter
in d dimensions — in d+1 dimensions
quantum — classical



$$S_A = \frac{1}{4G_N} \text{Area}(m_A)$$

m_A is the area of a minimal surface that hang from A into the d+1 dimensional bulk

This area is continuously connected to A.

Entropy of a black hole (Bekenstein)

$$S = \frac{\text{Area}}{4G_N}$$

Temperature of black hole: $T = \frac{\kappa}{2\pi}$

BH $\rightarrow$ thermal quantum CFT

- can count the microstates
- this gives the area law

The SYK model

$$H = \frac{1}{q!} \sum_{1 < \alpha \beta \gamma \delta < N} J_{\alpha\beta\gamma\delta}\, \chi_\alpha \chi_\beta \chi_\gamma \chi_\delta$$

Gaussian random

$$\langle J^2_{\alpha\beta\gamma\delta} \rangle = \frac{J^2 (q-1)!}{N^{q-1}}$$

χ_α are Majorana fermion

$$\{ \chi_\alpha, \chi_\beta \} = \delta_{\alpha\beta}$$

The same commutation relations as γ matrices $\quad \chi_\alpha \longrightarrow \gamma_\alpha$

Hilbert space

creation and annihilation operators from γ matrices

$$c_k = \frac{1}{\sqrt{2}} \left(\gamma_{2k} + i\gamma_{2k+1} \right) \qquad c_k^+ = \frac{1}{\sqrt{2}} \left(\gamma_{2k} - i\gamma_{2k+1} \right)$$

$k \neq l \qquad \{ c_k, c_l \} = 0 \qquad \{ c_k^+, c_l^+ \} = 0$

$$\{ c_k^+, c_l \} = 0$$

$$c_k c_k^+ + c_k^+ c_k =$$

$$\frac{1}{2} \left(\gamma_{2k} + i\gamma_{2k+1} \right) \left(\gamma_{2k} - i\gamma_{2k+1} \right) + \frac{1}{2} \left(\gamma_{2k} - i\gamma_{2k+1} \right) \left(\gamma_{2k} + i\gamma_{2k+1} \right)$$

$$= \frac{1}{2} \left(+\frac{1}{2} + \frac{1}{2} \right) - \cancel{\frac{i}{2} \gamma_{2k} \gamma_{2k+1}} + \cancel{\frac{i}{2} \gamma_{2k+1} \gamma_{2k}}$$

$$+ \frac{1}{2} \left(+\frac{1}{2} + \frac{1}{2} \right) + \cancel{\frac{i}{2} \gamma_{2k} \gamma_{2k+1}} - \cancel{\frac{i}{2} \gamma_{2k+1} \gamma_{2k}} = 1$$

This means that the c_k are the usual fermionic annihilation operators.

We can rewrite the Hamiltonian in terms of c_k and c_k^+. Then it is clear that the Hamiltonian does not conserve particle number.

So the Hilbert space is

$$|0\rangle$$
$$c_k^+|0\rangle$$
$$c_k^+c_e^+|0\rangle$$
$$\prod_{k=1}^{N/2} c_k^+|0\rangle$$

total number of states $\sum_{k=0}^{N}\binom{N/2}{k} = 2^{N/2}$.

Of couse this is just the dimension of γ-matrices in N dimensions.

The $q=2$ SYK model

Hamiltonian $H = i \sum_{i<j} J_{ij} \gamma_i \gamma_j$ ← Gaussian random

$H^\dagger = H$

$J_{ij} = -J_{ji}$

J_{ij} can be brought to standard form

$$J = U \begin{pmatrix} 0 & x_1 & & & 0 \\ -x_1 & 0 & & & \\ & & 0 & x_{i+1} & \\ 0 & & -x_{i+1} & 0 & \end{pmatrix} U^{-1}$$

$U \in O(2^{N/2})$

$$J_{ij}\gamma_i\gamma_j = U_{ip} J^D_{pq} U^{-1}_{qj} \gamma_i \gamma_j$$

$$= J^D_{pq} U_{ip}\gamma_i U^{-1}_{qj}\gamma_j$$

$$= J^D_{pq} \underbrace{U^T_{pi}\gamma_i}_{\substack{\text{equivalent set} \\ \text{of } \gamma \text{ matrices}}} U^T_{qj}\gamma_j$$

$$\{\underbrace{U_{pi}\gamma_i}_{\gamma_p'}, U_{qj}\gamma_j\} = U_{pi}U_{qj} \underbrace{\{\gamma_i, \gamma_j\}}_{\delta_{ij}}$$

$$= \delta_{pq}.$$

$$\Rightarrow \quad H = i \sum_{k=1}^{N/2} x_k \gamma'_{2k-1} \gamma'_{2k}$$

$$[\gamma_{2k-1}\gamma_{2k}, \gamma_{2l-1}\gamma_{2l}] = 0 \qquad k \neq l$$

$\Rightarrow$ the $\gamma_{2k-1}\gamma_{2k}$ can be diagonalized simultaneously

$$i\gamma_{2k-1}\gamma_{2k}\, i\gamma_{2k-1}\gamma_{2k} = (-)(-)\gamma_{2k-1}^2 \gamma_{2k}^2 = 1$$

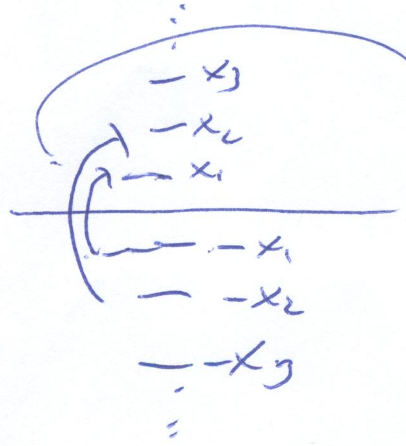
$\Rightarrow$ eigenvalues of H are given by

$$\sum_{k=1}^{N/2} s_k x_k \qquad s_k = \pm 1$$

<u>ground state</u>: all negative energy states are filled

$$E = -x_1 - x_2 \cdots - x_{\frac{N}{2}}$$

This is a state of $\frac{N}{2}$ particles



excited states: particle hole excitations

Fermi-level

So the $q=2$ SYK model is a Fermi liquid.

The $q=2$ Hamiltonian can also be rewritten in term of creation and annihilation operators

$$H = \sum_{k=1}^{N/2} x_k \left(2 c_k c_k^{\dagger} - 1 \right)$$

Thermodynamics of $q=2$ SYK model

$$Z = \sum_{n_k = 0,1} e^{-\beta \sum_{k=1}^{N/2} x_k (2 n_k - 1)}$$

$$= \prod_{k=1}^{N/2} \left(e^{\beta x_k} + e^{-\beta x_k} \right)$$

$$= \underbrace{\prod_{k=1}^{N/2} e^{\beta x_k}}_{e^{-\beta \sum_{k=1}^{N/2} (-x_k)}} \left(1 + e^{-2\beta x_k} \right)$$

$$\sum_k \to \int \rho(E)\, dE$$

$\rho(E)$: density of x_k

$e^{-\beta \sum_{k=1}^{N/2}(-x_k)}$: ground state energy.

$$\Rightarrow \log Z = -\beta E_0 + \int \rho(E)\, dE \log\left(1 + e^{-2\beta E} \right)$$

The level density is known from random matrix theory

$$\rho(E) = \frac{2N}{\pi} \sqrt{1 - E^2}$$

$$\int^{1} \rho(E)\, dE = \frac{2N}{\pi} \frac{\pi}{4} = \frac{N}{2}$$

$$\log 2 = -\beta E_0 + \frac{2N}{\pi}\int_0^1 \sqrt{1-E^2}\,\log\left(1+e^{-2\beta E}\right)dE$$

$$E = \sin\theta$$

$$= -\beta E_0 + \frac{2N}{\pi}\int_0^{\frac{\pi}{2}} \cos^2\theta \log\left(e^{-2\beta\sin\theta}+1\right)d\theta$$

low temperature limit $\beta \to 0$

$$\log 2 = -\beta E_0 + \frac{2N}{\pi}\int_0^{\frac{\pi}{2}} \underbrace{\cos^2\theta}_{1-\sin^2\theta}\left(1+e^{-2\beta\sin\theta}+\cdots\right)d\theta$$

cannot be neglected.

$$\simeq -\beta E_0 + \frac{2N}{\pi}\int_0^{\infty} \log\left(1+e^{-2\beta x}\right)dx$$

$$= -\beta E_0 + \frac{N}{\pi\beta}\frac{2\pi^2}{24}$$

zero temperature entropy is the constant term in the low temperature expansion. It vanishes as it should for a Fermi-liquid.

high temperature limit

$$\log 2 = -\beta E_0 + \underbrace{\frac{2N}{\pi}\frac{\pi}{4}\log 2}_{\log 2^{N/2}}$$