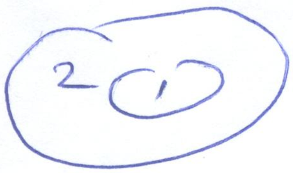
Srednicki's calculation

$$H = \frac{1}{2}\left(p_1^2 + p_2^2\right) + k_0\left(x_1^2 + x_2^2\right) + k_1\left(x_1 - x_2\right)^2$$

groundstate $\psi_0(x_1, x_2) = \frac{1}{\sqrt{\pi}} \frac{1}{(\omega_+\omega_-)^{1/4}} e^{-\frac{1}{2}(\omega_+ x_+^2 + \omega_- x_-^2)}$

$$x_\pm = \frac{x_1 \pm x_2}{\sqrt{2}} \qquad \omega_+ = k_0^{1/2}, \quad \omega_- = (k_0 + 2k_1)^{1/2}$$



choose x_2 as the outside

Density matrix

$$\rho_{out}(x_2, x_2') = \int_{-\infty}^{\infty} dx_1\, \psi_0(x_1, x_2)\, \psi_0^*(x_1, x_2')$$

$$= \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{\gamma - \beta}}\, e^{-\frac{\gamma}{2}(x_2^2 + x_2'^2) + \beta x_2 x_2'}$$

$$\beta = \frac{1}{4}\frac{(\omega_+ - \omega_-)^2}{\omega_+ + \omega_-} \qquad \gamma - \beta = \frac{2\omega_+\omega_-}{\omega_+ + \omega_-}$$

eigenvalue equation for ρ_{out}

$$\int_{-\infty}^{\infty} dx'\, \rho_{out}(x, x')\, f_n(x') = p_n f_n(x)$$

Then the entanglement entropy is

$$S = -\sum p_n \log p_n$$

$\sqrt{\omega_+\omega_-} = \sqrt{\gamma^2 - \beta^2}$

solution $p_n = (1 - \xi)\,\xi^n$

$$f_n = H_n(\alpha^{1/2} x)\, e^{-\alpha x^2/2}$$

Hermite polynomial

$$\xi = \frac{\beta}{\gamma + \alpha}$$

Then $S = -\sum_{n=1}^{\infty} (1-\xi)\, \xi^n \log(\xi^n \quad (1-\xi))$

$= -(1-\xi) \sum_{n=1}^{\infty} n\, \xi^n \log \xi - (1-\xi) \sum_{n=1}^{\infty} \xi^n \log(1-\xi)$

$= -(1-\xi) \log \xi \, \frac{d}{d\xi} \frac{1}{1-\xi} - \frac{(1-\xi)\,\xi}{1-\xi} \log(1-\xi)$

$= -(1-\xi) \log \xi \, \frac{1}{(1-\xi)^2} - \xi \log(1-\xi)$

$= - \frac{\log \xi}{1-\xi} - \xi \log(1-\xi)$

system of harmonic oscillators

$$H = \frac{1}{2} \sum p_i^2 + \frac{1}{2} \sum_{i,j=1}^{N} x_i K_{ij} x_j$$

ground state wave function

$$\psi_0(x_1, \dots, x_N) = \frac{1}{\pi^{N/4}} \frac{1}{\det(\Omega)} e^{-\frac{x \cdot \Omega \cdot x}{2}}$$

$$\Omega = \sqrt{K}$$

$$P_{out}(x_{n+1}, \dots x_N, x'_{n+1}, \dots x'_N)$$

$$= \int \prod_{i=1}^{n} dx_i \, \psi_0(x_1, \dots x_n, x_{n+1} \dots x_N) \times \psi_0^*(x_1 \dots x_n, x'_{n+1} \dots x'_N)$$

$\Omega^T = \Omega \qquad \Omega = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$

$$\rho_{out}(x, x') = e^{-\frac{1}{2}(x\gamma x + x'\gamma x') + x\beta x'}$$

$$\beta = \frac{1}{2} B^T A^{-1} B \qquad \gamma = C - \beta$$

To find the eigenvalues we use that

$\det G \; \rho_{out}(Gx, Gx')$ has the same

eigenvalues as ρ_{out}. Just change

integration variables $\quad d\,Gx' = \det G \, dx$

$$\gamma = V^T \gamma_D V$$ ← orthogonal; ↑ diagonalle

then $x = V^T \gamma_D^{-\frac{1}{2}} y$.

Then $\quad \rho_{out}(y, y') = e^{-\frac{1}{2}(y \cdot y + y' \cdot y') + y\beta' y'}$

$$\beta' = \gamma_D^{-\frac{1}{2}} V \beta V^T \gamma_D^{-\frac{1}{2}}$$

no we can diagonalize β' with eigenvalues β_i'

$$\Rightarrow \quad \rho_{out}(z, z') \sim e^{-\frac{1}{2}\sum (z_i^2 + z_i'^2) + \sum_i \beta_i' z_i z_i'}$$

This is the same as we had for the two coupled oscillators with

$\gamma \to 1$ and $\beta \to \beta_i'$

eigenvalue equation

$$\prod_i \int dz_i' \, e^{-\frac{1}{2}(z_i^2 + z_i'^2) + \beta_i'(z, z_i')} \, f_{n_i}(z_i')$$

$$p_{\vec{n}} = \prod_i \xi_i^{n_i}(1-\xi_i), \quad \xi_i = \frac{\beta_i'}{1+(1-\beta_i'^2)^{1/2}}$$

$$S = -\sum_{\{n_i\}} \prod_i p_{n_i} \log p_{n_i}$$

sum of product = product of sum.

$$\Rightarrow S = \sum_i S(\xi_i)$$

<u>Quantum field theory</u>

$$H = \frac{1}{2}\int d^3x \left(\Pi^2(x) + (\vec{\nabla}\varphi)^2\right)$$

Expand the field in spherical harmonics

$$\varphi_{lm} = x\int d\Omega \, Z_{lm}(\theta,\varphi)\, \varphi(\vec{x})$$

$$\Pi_{lm} = x\int d\Omega \, Z_{lm}(\theta,\varphi)\, \Pi(\vec{x})$$

$x = |\vec{x}|$

$Z_{lm} = \sqrt{2}\, \mathrm{Re}\, Y_{lm}, \; m>0 \qquad Z_{l0} = Y_{l0}$

$\;\;\;\;\; = \sqrt{2}\, \mathrm{Im}\, Y_{lm}, \; m<0$

$$[\varphi_{lm}(x), \Pi_{l'm'}(x')] = i\,\delta_{ll'}\delta_{mm'}\delta(x-x')$$

radial coordinate x is replaced by a lattice with spacing a

a $\quad x \to \quad (N+1)a = L$

This give the radial Hamiltonian

$$H_{\ell m} = \frac{1}{2a} \sum_{j=1}^{N} \Pi_{\ell m, j}^{2} + \left(j+\tfrac{1}{2}\right)^{2} \left(\frac{\varphi_{\ell m, j}}{j} - \frac{\varphi_{\ell m, j+1}}{j+1} \right)^{2} + \frac{\ell(\ell+1)}{j^{2}} \varphi_{\ell m, j}^{2}$$

$$[\varphi_{\ell m j}, \Pi_{\ell' m' j'}] = i\, \delta_{\ell\ell'} \delta_{m m'} \delta_{j j'}$$

$$\varphi_{\ell m, N+1} = 0$$

Note that before discretization, the Hamiltonian is given by

$$H = \sum H_{\ell m}$$

$$H_{\ell m} = \frac{1}{2} \int_{0}^{\infty} dx \left(\Pi_{\ell m}^{2}(x) + x^{2} \left(\frac{\partial}{\partial x} \frac{\varphi_{\ell m}}{x} \right)^{2} + \frac{\ell(\ell+1)}{x^{2}} \varphi_{\ell m}^{2}(x) \right)$$

this is just the radial Hamiltonian.

The inside is now the first n lattice site.

$$\Psi_0 = \prod_{lm} \Psi_{0lm}$$

$$\Rightarrow \rho = \prod_{lm} \rho_{lm}$$

eigenvalues of ρ : $\prod_{lm} P^{lm}_{n_{lm}}$

$$S = \sum_{\{n_{lm}\}} -\prod_{lm} P^{lm}_{n_{lm}} \log \prod_{lm} P^{ln}_{n_{lm}}$$

sum of product = product of sum

So $\quad S(n, N) = \sum_{lm} S_{lm}(n, N)$

Hamiltonian does not depend on m

$$\Rightarrow S(n, N) = \sum_{l} (2l+1) S_{l\,m=0}(n, N)$$

We choose $l \gg N$ then the $l(l+1)$ term in the Hamiltonian dominates

Then $\quad S_l(n, N) = \xi_l(n) \left(-\log \xi_l(n) + 1\right)$

$$\xi_{ln} = \frac{n(n+1)(2n+1)^2}{64 l^2 (l+1)^2} + \mathcal{O}\left(\frac{1}{l^6}\right)$$

$$n = \frac{R}{a}$$

$$S = \sum_\ell (2\ell+1)\, S_\ell(n,N)$$

$$= \frac{4n^4}{64} \sum_\ell \frac{(2\ell+1)}{\ell^2(\ell+1)^2}\left(1 - \log \frac{4n^4}{64\,\ell^2(\ell+1)^2}\right)$$

this is $\sim R^4$.

To see the true R dependence, the sums have to be evaluated numerically.

Bombelli, Koul, Lee, Sorkin, PRD 1986

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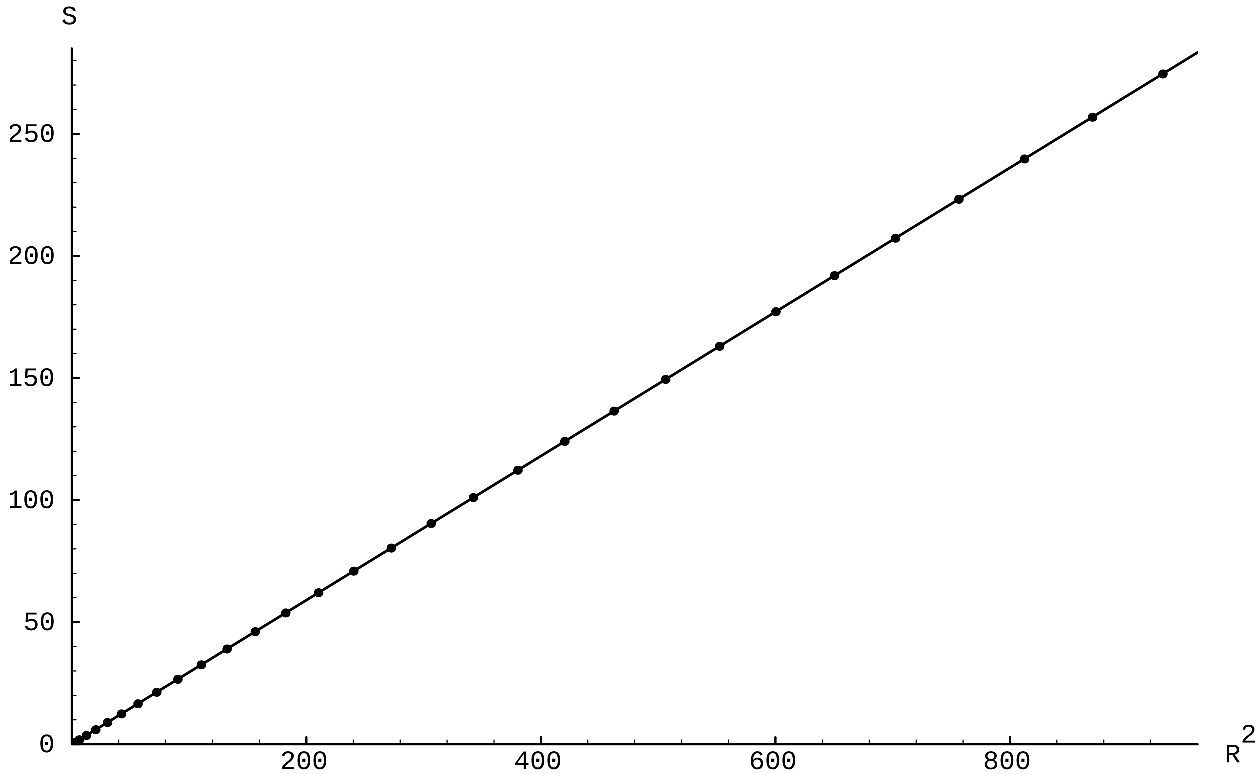

Fig. 1. The entropy S resulting from tracing the ground state of a massless scalar field over the degrees of freedom inside a sphere of radius R. The points shown correspond to regularization by a radial lattice with $N = 60$ sites; the line is the best linear fit. R is measured in lattice units, and is defined to be $n + \frac{1}{2}$, where n is the number of traced sites.