# Entanglement entropy for a two-spin system

see Thomas Hartman, Lecture 18

$\uparrow\downarrow$ $\quad$ $\uparrow\downarrow$
A $\quad$ B $\quad$ each one qubit (spin)

states of the full system $|\uparrow\downarrow\rangle_{AB}$ $\;|\downarrow\uparrow\rangle_{AB}$ $\;|\uparrow\uparrow\rangle_{AB}$ $\;|\downarrow\downarrow\rangle_{AB}$

$$|i\,j\rangle = |i\rangle_A \otimes |j\rangle_B$$

state of the full systen $|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)$

density matrix of the full system

$$\rho = |\psi\rangle\langle\psi|$$

The total Hilbert space is four dimensional, so $\rho$ is a $4\times 4$ matrix

$$\rho = \tfrac{1}{2}\left( |\uparrow\downarrow\rangle\langle\uparrow\downarrow| + |\uparrow\downarrow\rangle\langle\downarrow\uparrow| + |\downarrow\uparrow\rangle\langle\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle|\downarrow\uparrow\rangle\right)$$

$$= \begin{array}{c|c} & |\uparrow\uparrow\rangle \quad |\uparrow\downarrow\rangle \quad \langle\downarrow\uparrow| \quad |\downarrow\downarrow\rangle \\ \begin{array}{c} |\uparrow\uparrow\rangle \\ |\uparrow\downarrow\rangle \\ |\downarrow\uparrow\rangle \\ |\downarrow\downarrow\rangle \end{array} & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

entanglement spectrum $\left(\frac{1}{2}-\lambda\right)^2 - \frac{1}{4} = 0$

$$\Rightarrow \lambda = 0,\ 1$$

so it is a pure state

Next we calculate the reduced density

$$\rho_A = {}_B\langle\uparrow|\rho|\uparrow\rangle_B + {}_B\langle\downarrow|\rho|\downarrow\rangle_B$$

$$= \frac{1}{2}\, {}_B\langle\uparrow|\,|\uparrow\downarrow\rangle\langle\uparrow\downarrow|\,|\uparrow\rangle_B + \frac{1}{2}\, {}_B\langle\downarrow|\,|\downarrow\uparrow\rangle\langle\downarrow\uparrow|\,|\downarrow\rangle_B$$

the other two combinations give zero

$$= \frac{1}{2}\, |\downarrow\rangle_A\langle\downarrow|_A + \frac{1}{2}\, |\uparrow\rangle\langle\uparrow|_A$$

$$= \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

entanglement entropy

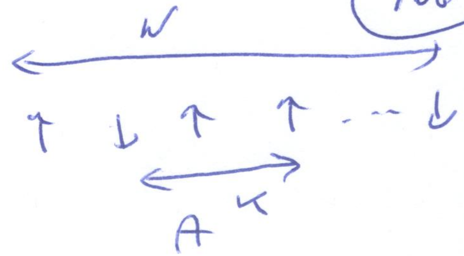
$$S = -\operatorname{Tr} \rho \log \rho$$

$$= -2\, \frac{1}{2} \log \frac{1}{2} = \log 2$$

if $\rho_A$ is proportional to the identity, we say that $\rho_A$ is maximally mixed and the initial state is maximally entangled.

The entanglement entropy counts the number of entangled states as $e^S$ in the full system that is in a pure state.

Second example




N spins

A has K spins

B has N-K spin

pure state of the full system

$$\Psi = \sum c_{K_1 \ldots K_N} |K_1\rangle \ldots |K_N\rangle$$

with $|K_j\rangle$ spin up or spin down

We now choose $c_{K_1 \ldots K_N}$ Gaussian random with zero average and $\sigma^2 = 2^{-N}$

normalization of $\Psi$

$$\langle \Psi | \Psi \rangle = \underbrace{\sum_{K_1 \ldots K_N} |c_{K_1 \ldots K_N}|^2}_{2^N \text{ term}} = 1$$

average density matrix

$$|\Psi\rangle\langle\Psi| = \sum_{K_1 \ldots K_N} 2^{-N} |K_1 \ldots K_N\rangle \langle K_1 \ldots K_N|$$

Next we calculate $\rho_A$ and choose A the first p spins

$$\rho_A = \mathrm{Tr}_B |\Psi\rangle\langle\Psi| .$$

$$= \sum_{K_1 \ldots K_p} 2^{-N} \, 2^{N-p} |K_1 \ldots K_p\rangle \langle K_1 \ldots K_p|$$

$$= 2^{-p} \sum |K_1 \ldots K_p\rangle \langle K_1 \ldots K_p|$$

entanglement entropy

$$S_A = -\mathrm{Tr}\ \rho_A \log \rho_A$$
$$= -2^p\ 2^{-p} \log 2^{-p} = p \log 2$$

when we identify $p$ as the volume of the system, than the entanglement entropy is proportianal to the volume of the system. This is called the volume law.

Third example

$$|\Psi\rangle = 2^{-N/2} \sum_{k_1 \cdots k_N} |k_1 \cdots k_N\rangle$$

$$\langle\Psi|\Psi\rangle = 1$$

$$\rho = 2^{-N} \sum_{\substack{k_1 \cdots k_N \\ l_1 \cdots l_N}} |k_1 \cdots k_N\rangle\langle l_1 \cdots l_N|$$

choose $A$ again as the first $p$ spins

Then $\rho_A = \mathrm{Tr}_B\, \rho = 2^{-N}\ 2^{N-p} \underbrace{\sum_{\substack{k_1 \cdots k_p \\ l_1 \cdots l_p}} |k_1 \cdots k_p\rangle\langle l_1 \cdots l_p|}$

$$S_A = -\mathrm{Tr}\ \rho_A \log \rho_A$$
$$= -2^{-p}\ 2^p \log 2^p$$

matrix with all matrix elements 1 eigenvalues $2^p$ and

## Fourth example

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\cdots\uparrow\rangle + |\downarrow\cdots\downarrow\rangle\right)$$

$$\langle\psi|\psi\rangle = 1$$

then

$$\rho = \frac{1}{2}\Big(|\uparrow\cdots\uparrow\rangle\langle\uparrow\cdots\uparrow| + |\uparrow\cdots\uparrow\rangle\langle\downarrow\cdots\downarrow| + |\downarrow\cdots\downarrow\rangle\langle\uparrow\cdots\uparrow| + |\downarrow\cdots\downarrow\rangle\langle\downarrow\cdots\downarrow|\Big)$$

A is first p spins and B is the rest

$$\rho_A = \mathrm{Tr}_B\,\rho = \frac{1}{2}\left(|\uparrow\cdots\uparrow\rangle_p\langle\uparrow\cdots\uparrow|_p + |\downarrow\cdots\downarrow\rangle_p\langle\downarrow\cdots\downarrow|_p\right)$$

entanglement entropy

$$S_A = -2\,\frac{1}{2}\log\frac{1}{2} = \log 2$$

This scales with the area of the system which is just one point.
This is the area law which is often found for the ground state of a system which is particular.

There is a theorem due to Hastings that the ground state of a one-dimensional gapped system with a local Hamiltonian has an area law.

Of course this means that $S_A$ is constant.

Entanglement entropy of A and its complement are the same.

state of full system $|\Psi\rangle = \sum \underset{\substack{\uparrow \\ \text{rectangular} \\ \text{matrix}}}{c_{nN}} \underbrace{|n\rangle}_{A} \otimes \underbrace{|N\rangle}_{\bar{A}}$

$$\rho = |\Psi\rangle\langle\Psi|$$

$$\rho_A = \mathrm{Tr}_{\bar{A}}\,\rho \qquad \rho_{\bar{A}} = \mathrm{Tr}_A\,\rho$$

$$\rho_A = \sum c_{nN} c^*_{n'N'} \mathrm{Tr}_N |n\rangle|N\rangle\langle n'|\langle N'|$$

$$= \sum_N \underbrace{c_{nN} c^+_{Nn'}} |n\rangle\langle n'|$$

entanglement spectrum is given by the eigenvalues of $cc^+$

$$\rho_{\bar{A}} = \sum c_{nN} c^*_{n'N'} \mathrm{Tr}_n |n\rangle|N\rangle\langle n'|\langle N'|$$

$$= \sum_n c_{nN} c^*_{n'N'} \cdot |N\rangle\langle N'| \delta_{nn'}$$

$$= \sum_n c_{nN} c^*_{nN'} |N\rangle\langle N'|$$

$$= \sum_n c^+_{N'n} c_{nN} |N\rangle\langle N'|$$

entanglement spectrum is given by the eigenvalues of $c^+c$

$c^+c$ and $c\,c^+$ have the same nonzero eigenvalues.

Proof: Singular value decomposition which is valid for any matrix

$$c = U \lambda V^{-1} \qquad \lambda_k \geq 0$$

diagonal

$$\Rightarrow \quad c^+c = V \lambda^2 V^{-1}$$

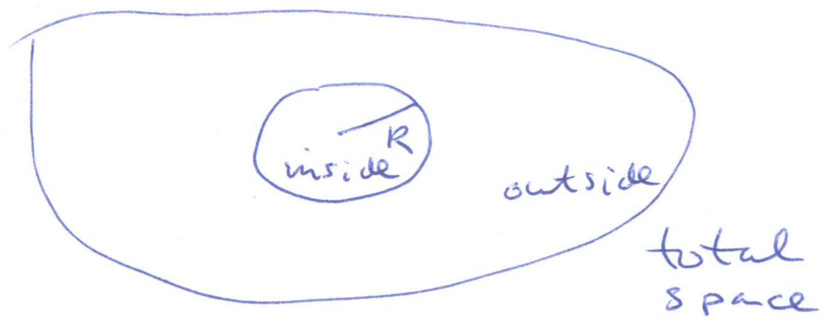
$$c\,c^+ = U \lambda^2 U^{-1} \qquad \text{qed}$$

when $c$ is rectangular

$c^+c$ and $c\,c^+$ have a different number of eigenvalues. The additional eigenvalues are zero.

Next we discuss the paper by Mark Srednicki, Entropy and area, PRL 71 (1993) 666.

- massless scalar field theory
- ground state $|0\rangle$
- density matrix $|0\rangle\langle 0| \equiv \rho$




$$\rho_{out} = \mathrm{Tr}_{inside}\, \rho$$

$$\rho_{inside} = \mathrm{Tr}_{outside}\, \rho$$

$$S_{out} = -\mathrm{Tr}\, \rho_{out} \log \rho_{out}$$

We expect that $S_{out} \sim V \sim R^d$ because the entropy is extensive

$$S_{in} = -\mathrm{Tr}\, \rho_{in} \log \rho_{in}$$

We have that the entanglement of a system and its complement are the same $\Rightarrow$ $S_{in} = S_{out}$

The only thing that the inside and the outside have in common is their boundary

$\Rightarrow \quad S \sim A \sim R^{d-1}$

entropy should be dimensionless.
We have a uv scale $\Lambda^{-1}$, this is the cut-off eg the lattice spacing and the infra-red scale which is the size of the system $L$.

When we have a local Hamiltonian, the properties should not depend on $L$.
So we must have

$$S = \kappa \Lambda^2 A$$

$\uparrow$ constant

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