3. Fluctuation dissipation theorem

magnetization density $m(r)$

correlation function $\Gamma(r) = \langle m(r)\, m(0)\rangle - \langle m(r)\rangle\langle m(0)\rangle$

typical form $\Gamma(r) \sim r^{-p}\, e^{-r/\xi}$

fluctuation-dissipation theorem

For a translation invariant system we have

$$\chi = \frac{1}{V}\frac{\partial M}{\partial H} = \frac{1}{kT}\int d^3r\, \Gamma(r)$$

↑ magnetic susceptibility

Proof: $\mathcal{H} = H_0 - \int H\, m(r)\, d^3r$

$$\chi = \frac{1}{V}\partial_H \frac{1}{Z}\mathrm{Tr}\int m(r)\, e^{-\beta\mathcal{H}}\, d^3r$$

$$= \frac{\beta}{V}\frac{1}{Z}\mathrm{Tr}\int d^3r\, d^3r'\, m(r)\, m(r')\, e^{-\beta\mathcal{H}}$$

$$- \frac{\beta}{V}\frac{1}{Z^2}\mathrm{Tr}\int d^3r\, m(r)\, e^{-\beta\mathcal{H}}\,\mathrm{Tr}\int d^3r'\, m(r')\, e^{-\beta\mathcal{H}}$$

$$= \beta\int d^3r\,\langle m(r)\, m(0)\rangle - \beta\int d^3r\,\langle m(r)\rangle\langle m(0)\rangle$$

$$= \beta\int d^3r\, \Gamma(r)$$

4a) Critical Exponent

dimensionless temperature $t = \frac{T - T_c}{T_c}$

Field $h = \frac{H}{kT}$

heat capicity $c \sim t^{-\alpha}$, $T < T_c$

$\sim t^{-\alpha'}$, $T > T_c$

magnetization $M \sim |t|^{\beta}$, $T < T_c$

susceptibility $\chi = \frac{1}{V} \partial_H M \sim |t|^{-\gamma}$, $T < T_c$

Equation of state $M \sim H^{1/\delta}$, $T = T_c$

Correlation function $\Gamma(r) \sim r^{-p} e^{-r/\xi}$ for $T \to T_c$

$$p \equiv d - 2 + \eta$$
$$\xi \sim |t|^{-\nu}$$

	Exp	Ising 3d
α	0-0.14	0.12
β	0.32-0.39	0.31
γ	1.3 - 1.4	1.25
δ	4 - 5	5
ν	0.6-0.7	0.64
η	0.05	0.05

There are relations between the critical exponents

e.g. Widom relation $\gamma = \beta(\delta - 1)$

exp $1.33 = 0.35 \times 3.5 = 1.23$

3d Ising $1.25 = 0.31 \times 4 = 1.24$

4b) Scaling

idea at the critical point there is only one scale: the correlation length ξ

$$\Gamma(r) \sim r^{-(d-2+\eta)} e^{-r/\xi}$$

$$\Gamma(r) = \langle m(r)\, m(0) \rangle - \langle m(r) \rangle \langle m(0) \rangle$$

$$\Rightarrow \int d^d r\, \Gamma(r) \sim M^2 \xi^d$$

$$\xi^{2-\eta} \Rightarrow \Gamma \sim \xi^{-\frac{d-2+\eta}{2}}$$

$$\chi \sim \int d^d r\, \Gamma(r) \Rightarrow \chi \sim \xi^{2-\eta}$$

Free energy $\times \beta$ = dimensionless

$$\Rightarrow \frac{F\beta}{V} \sim \xi^{-d}$$

specific heat $C = T^2 \frac{\partial^2 F}{\partial T^2}$

$$\xi \sim |t|^{-\nu} \Rightarrow C \sim |t|^{\nu d - 2}$$

relations $\beta = \nu(d-2+\eta)/2$

$\gamma = \nu(2-\eta)$ Fisher

$-\alpha = \nu d - 2$ Josephson

(or hyperscaling because d is a parameter)

Next we derive a relation for δ when $T \to T_c$ and H is small

in the partition function H occurs as

$$\int d^d r \, H M$$

$$T \to T_c \qquad M \sim \xi^{-(d-2+\eta)/2}$$
$$d^d r \sim \xi^d$$

to stay at constant physics we need $H \sim \xi^{-\frac{d}{2}+\frac{\eta}{2}-1}$

but close to T_c $\quad M \sim H^{\frac{1}{\delta}}$

$$\Rightarrow \quad \underbrace{\xi^{-\frac{d-2+\eta}{2}}}_{-\frac{\beta}{\nu}} \sim \xi^{\frac{1}{\delta}\left(-\frac{d}{2}+\frac{\eta}{2}-1\right)}$$

$$\Rightarrow \quad \beta\delta = \nu(2+d-\eta)/2$$

$\exists$ 4 relations $\Rightarrow$ we have only two independent exponents

The phase of the system also depends on two parameters, H and T. Is there a relation?