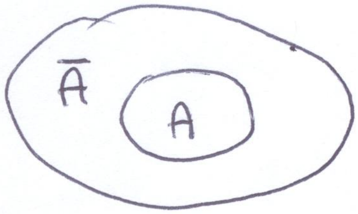
22. Entanglement entropy



$$\mathcal{H} = H_A \otimes H_{\bar{A}}$$

$$H\Psi = E\Psi$$

A state of the subsystem is not described by a single state Ψ_A of H_A such that for operators O_A

$$\langle \Psi_A' | O_A | \Psi_A \rangle = \langle \Psi | O_A \otimes 1 | \Psi \rangle$$

What we need are mixed states $\{|\Psi_i\rangle P_i\}$ with P_i the probability to find $|\Psi_i\rangle$

Then $\langle O \rangle = \sum_i P_i \langle \Psi_i | O | \Psi_i \rangle$

We will next show that for any stat of the full system, Ψ, we have

$$\langle \Psi | O_A \times 1 | \Psi \rangle = \sum P_i \langle \Psi_i^A | O | \Psi_i^A \rangle$$

$P_i \geq 0$ ↑ states of system A

To do that we will use the density matrix

$$\rho_A = \sum_i P_i |\Psi_i^A\rangle\langle\Psi_i^A|$$

corresponding to the mixed state

Then $\langle O_A \rangle = \mathrm{Tr}\, O_A \rho_A = \sum_i P_i \langle \Psi_i^A | O | \Psi_i^A \rangle$

Let us consider a state of the full system

$$|\psi\rangle = \sum c_{nN} |\psi_n\rangle \otimes |\psi_N\rangle \qquad \sum_{nN} |c_{nN}|^2 = 1$$

Then $|\psi\rangle\langle\psi|$ is the density matrix of the full system.

\rho = |\psi><\psi|

The ρ_A is given by $\rho_A = \mathrm{Tr}_{\bar{A}}\, \rho$

$$\rho_A = \mathrm{Tr}_{\bar{A}} \sum c_{nN'} c^*_{mN}\, |\psi_n\rangle \otimes |\psi_{N'}\rangle\langle\psi_N| \otimes \langle\psi_m|$$

$$= \sum_{\substack{nm \\ N}} c_{nN} c^*_{mN}\, |\psi_n\rangle\langle\psi_m|$$

$$\mathrm{Tr}(O_A \rho_A) = \sum_{\substack{nm \\ N}} \langle\psi_m| O_A |\psi_n\rangle\, c_{nN} c^*_{mN}$$

$$\sum_M \langle\psi_M | 1 | \psi_N\rangle \qquad M$$

$$= \sum_{\substack{NM \\ nn}} \langle\psi_m| \otimes \langle\psi_M|\; O_A \otimes 1\; |\psi_n\rangle \otimes |\psi_N\rangle\; c_{nN} c^*_{mM}$$

$$= \langle\psi|\; O_A \otimes 1\; |\psi\rangle$$

- ρ_A is called the reduced density matrix
- tracing over $\bar{A}$ is called a partial trace

entropy of a state

$$S = -\sum p_i \log p_i = -\mathrm{tr}\, \rho_A \log \rho_A$$

(von Neumann entropy) $\uparrow$ eigenvalues of ρ_A

microcanonical ensemble

$$p_i = \frac{1}{n} \quad \text{the} \quad S = -\sum \frac{1}{n} \log \frac{1}{n}$$
$$= \log n$$

canonical ensemble

$$\left\{ |E_i\rangle, p_i = \frac{e^{-\beta E_i}}{Z} \right\}$$

Entanglement:

When a subsystem is not entangled we can write

$$|\psi\rangle = |\psi_A\rangle \otimes |\psi_{\bar{A}}\rangle$$

When a system is entangled this is not possible.

Entanglement entropy

$$S = -\rho_A \,\mathrm{Tr} \log \rho_A$$

Renyi entropy

$$S_\alpha = \frac{1}{1-\alpha} \log \sum_i p_i^\alpha$$
$$= \frac{1}{1-\alpha} \log \mathrm{Tr}\, \rho^\alpha$$

$\mathrm{Tr}\,\rho^\alpha$ for integer α is easier to calculate

then $\mathrm{Tr}\log\rho^\alpha$

$$\lim_{\alpha\to 1} S_\alpha = \lim_{\alpha\to 1}\frac{1}{1-\alpha}\log\sum_i p_i^\alpha$$

$$p_i^\alpha = p_i\, e^{(\alpha-1)\log p_i} = p_i + (\alpha-1)\log p_i$$

$$\sum p_i = 1$$

$$= -\sum_i p_i \log p_i$$

The p_i are known as the entanglement spectrum

pure state of $A\cup B$ if ρ_A is the density matrix of A

Then a pure state is

$$|\psi\rangle = \sum_i \sqrt{p_i}\,|\psi_i^A\rangle\otimes|\psi_i^B\rangle$$

(purification)

orthogonal state of B

this is a pure state of the combined system

$$\langle\psi|O_A\otimes 1|\psi\rangle$$

$$= \sum_i p_i \langle\psi_i^A|O_A|\psi_i^A\rangle$$

Thermo Field double state $\frac{1}{2} \sum e^{-\beta E_i/2} |E_i\rangle \otimes |E_i\rangle$

Note that the density matrix of A and B are the same by taking the ~~reduce~~ trace.

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