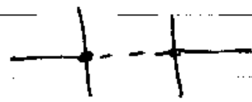
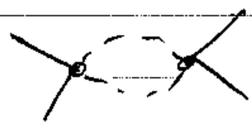
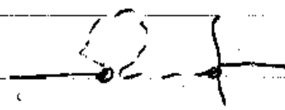
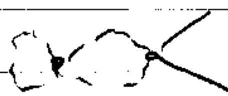
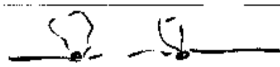
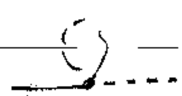
- A gives only disconnected diagrams
- The total # of σ-legs should be even

			combinatorial factor
BB		$O(S^4)$	$4.4 = 16$
CC		$O(S^4)$	$\binom{4}{2}\binom{4}{2}2 = 72$
BD		$O(S^4)$	$\binom{4}{2}2,4 = 48$
EC		$O(S^2)$	$\binom{4}{2}\binom{4}{2}2.2 =$????
EE1		$O(S^0)$	$4.3.2 = 24$
EE2		$O(S^0)$	$\binom{4}{2}\binom{4}{2}2 = 72$
DD1		$O(S^2)$	$4.4.3.3 = 144$
DD2		$O(S^2)$	$4.4\,3.2 = 48$

DD1 = 0  $\sim \delta(k_1+k_2)$

k_1 k_2

$|k|<\frac{\Lambda}{e}$ $\uparrow\ |k| > \frac{\Lambda}{e}$ but momentum is conserved in loop

BD = 0 

$|k|<\frac{\Lambda}{e}$ $\nwarrow\ |k| > \frac{\Lambda}{e}$

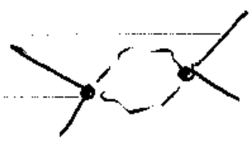
EE1 and EE2 don't contain S-field and just result in a constant factor

BB is $\mathcal{O}(S^6)$; we only keep terms up to order S^4

We keep only ε corrections to u_2

EE and DD_2 are $\mathcal{O}(u_0^2)$ but $u_0^* = \mathcal{O}(\varepsilon)$
(we should check this)

we can neglect them.

The only remaining diagram is 

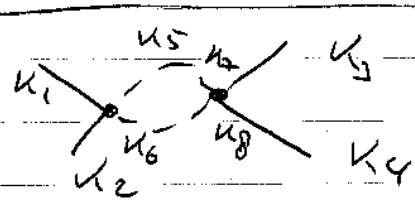
(22e) RG equations

$u_{2\ell} = z^2 \ell^{-d-2} (r_0 \ell^2 + k^2 + 12 \ell^2$  $)$

$u_{4\ell} = z^4 \ell^{-3d} \big[$  $-\ 36$  $\big]$

minus sign because $u_{4\ell}$ occurs with minus sign in exponent

(22f) Calculation of CC

CC = 

$$CC = \left(\frac{u_0}{4}\right)^2 \int_0^{\Lambda/\ell} dk_1 \dots dk_4 \int_{\Lambda/\ell}^{\Lambda} dk_5 \dots dk_8 \, \hat{S}'_\ell(k_1) \dots \hat{S}'_\ell(k_4)$$

$$\times (2\pi)^d \delta(k_1+k_2+k_5+k_6) \times (2\pi)^d \delta(k_3+k_4+k_7+k_8)$$

$$\times (2\pi)^d \frac{\delta(k_5+k_7)}{r_0+k_5^2} (2\pi)^d \frac{\delta(k_6+k_8)}{r_0+k_8^2}$$

- Do k_7 and k_8 integrations.

New delta function $\delta^d(k_1 + k_2 + k_5 + k_6)$

$\therefore \delta^d(k_3 + k_4 - k_5 - k_6)$

Do k_6 integration $\Rightarrow$ only $\delta^d(k_1 + k_2 + k_3 + k_4)$ remains

$$\Rightarrow c = \left(\frac{u_0}{4}\right)^2 \int_0^{\Lambda/\ell} dk_1 \ldots dk_4 \; \hat{S}'(k_1) \ldots \hat{S}'(k_4) \; \delta^d(k_1 \ldots k_4)$$

$$\times (2\pi)^d \int_{\Lambda/\ell}^{\Lambda} dk_5 \; \frac{1}{r_0 + k_5^2} \; \frac{1}{r_0 + (k_3 + k_4 - k_5)^2}$$

To $O(\varepsilon)$ the dependence of the coefficient on k_3 and k_4 can be neglected. We put them to zero

normalization condition: coeff of k^2 is $\frac{1}{2}$

$\Rightarrow z^2 \ell^{-d-2} = 1$

$$\Rightarrow r' = (r_0 \ell^2 + 12 \frac{u_0}{4} \ell^2 I_1)$$

$$u' = u_0 \, z^4 \ell^{-3d} (1 - 9 u_0 I_2)$$

$$\ell^{(2+d)2-3d} = \ell^{4-d} = \ell^{\varepsilon}$$

$$I_1 = \int_{\Lambda/\ell}^{\Lambda} \frac{dk}{r_0 + k^2} \qquad I_2 = \int_{\Lambda/\ell}^{\Lambda} dk \; \frac{1}{(r_0 + k^2)^2}$$

$$\sim \Lambda^{d-2} \qquad\qquad \sim \Lambda^{d-4}$$

Calculation of I_1 and I_2

$$I_1 = \int_{\Lambda/e}^{\Lambda} dk \frac{1}{r_0 + k^2} = \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})} \int_{\Lambda/e}^{\Lambda} \frac{k^{d-1}\,dk}{(2\pi)^d} \left(\frac{1}{k^2} - \underbrace{\frac{r_0}{k^4} + \dots}_{\text{neglect}} \right.$$

$$= \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})} \left(\frac{1}{d-2} \left(\Lambda^{d-2} - \left(\frac{\Lambda}{e}\right)^{d-2} \right) \right.$$

$$\left. - \frac{r_0}{d-4} \left(\Lambda^{d-4} - \left(\frac{\Lambda}{e}\right)^{d-4} \right) \right)$$

$$\frac{e^{(d-4)\log\Lambda} - e^{(d-4)\log\frac{\Lambda}{e}}}{} \simeq (d-4)\log \ell$$

$$\Rightarrow I_1 = \frac{(2\pi)^{d/2}}{(2\pi)^d\,\Gamma(d/2)} \left(\frac{1}{2} \left(\Lambda^2 - \frac{\Lambda^2}{e^2} \right) - r_0 \log \ell \right.$$

$$\frac{1}{d-2} = \frac{1}{2+d-4} = \frac{1}{2} + \underbrace{\frac{(4-d)}{4}}_{\text{neglect } (*)}$$

$$I_2 = \int_{\Lambda/e}^{\Lambda} dk \frac{1}{(r_0+k^2)^2} = \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})(2\pi)^d} \int_{\Lambda/e}^{\Lambda} k^{d-1} dk \left(\frac{1}{k^4} - \underbrace{\frac{2r_0}{k^6}}_{\text{vanishes for } \Lambda \to \infty} \right)$$

$$= \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})(2\pi)^d} \frac{1}{d-4} \left(\Lambda^{d-4} - \left(\frac{\Lambda}{e}\right)^{d-4} \right.$$

This also justifies the neglect of k_4 and k_5

$$= \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})(2\pi)^d} \log \ell$$

(*) $u_0 \sim O(\varepsilon)$

This follows from the analysis of the renomalization group equations

Recursion relations

$$r' = \ell^2 \Big(r_0 + \frac{3}{4\pi^2} \frac{u_0}{4} \Lambda^2 \left(1 - \frac{1}{\ell^2}\right)$$

$$- \frac{3}{2\pi^2} \frac{u_0}{4} r_0 \log \ell \Big)$$

for $d \to 4$

$$\frac{u'}{4} = \ell^{\varepsilon} \left(\frac{u_0}{4} - \frac{9}{2\pi^2} \left(\frac{u_0}{4}\right)^2 \log \ell \right)$$

$\uparrow$

$\ell^{\varepsilon} \quad \mathcal{O}(\varepsilon^2)$

$e^{\varepsilon \log \ell}$ we can put d in other

$= 1 + \varepsilon \log \ell$ factor $d = 4$

$$\Rightarrow \frac{u'}{4} = \frac{u_0}{4} + \varepsilon \frac{u_0}{4} \log \ell - \frac{9}{2\pi^2} \left(\frac{u_0}{4}\right)^2 \log \ell$$

22e) Analysis of the RG equations

Fixed points:

Gaussian fixed point $u^* = r^* = 0$

Wilson-Fisher fixed point $u^* = \frac{4 \cdot 2\pi^2}{g}\varepsilon$

$$\Rightarrow r^* = \ell^2\Big(r^* + \frac{3}{4\pi^2}\frac{2\pi^2}{g}\varepsilon\,\Lambda^2\Big(1-\frac{1}{\ell^2}\Big) - \frac{3}{2\pi^2}\underbrace{\frac{2\pi^2}{g}\varepsilon\, r^*}_{O(\varepsilon^2)\text{ neglect}}\log\ell\Big)$$

$$\Rightarrow r^* = -\frac{\varepsilon\Lambda^2}{6}$$

linearized RG equations

$$M = \begin{pmatrix} \frac{\partial r'}{\partial r} & \frac{\partial r'}{\partial u} \\ \frac{\partial u'}{\partial r} & \frac{\partial u'}{\partial u} \end{pmatrix} = \begin{pmatrix} \ell^2 - \frac{3u_0}{8\pi^2}\ell^2\log & \frac{3\Lambda^2}{16\pi^2}(\ell^2-1) - \frac{3 r_0}{8\pi^2}\log\ell \\ 0 & 1+\varepsilon\log\ell - \frac{9}{8\pi^2}2u_0\log\ell \end{pmatrix}$$

$$M_{Gaussian} = \begin{pmatrix} \ell^2 & \frac{3\Lambda^2}{16\pi^2}(\ell^2-1) \\ 0 & 1+\varepsilon\log\ell \end{pmatrix}$$

$$M_{WF} = \begin{pmatrix} \ell^2\big(1-\frac{\varepsilon}{3}\log\ell\big) & \frac{3\Lambda^2}{16\pi^2}(\ell^2-1) + \frac{\varepsilon\Lambda^2}{16\pi^2}\log\ell \\ 0 & 1-\varepsilon\log\ell \end{pmatrix}$$

Gaussian fixed point

$$\lambda_t = \ell^2 \Rightarrow y_t = 2$$
$$\Lambda_u = \ell^\varepsilon \Rightarrow y_u = \varepsilon$$

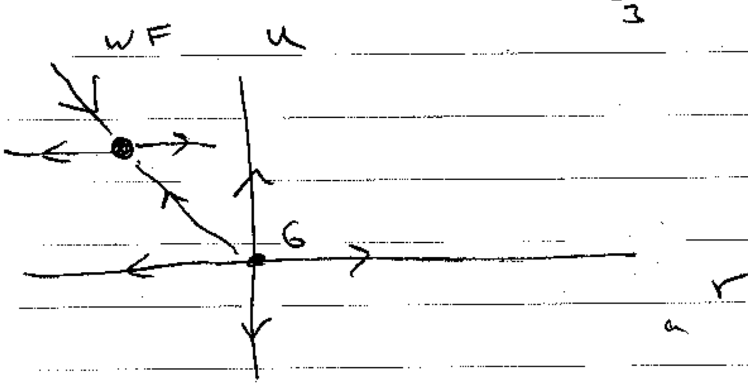
WF fixed point $\Lambda_t = \ell^{2-\frac{\varepsilon}{3}} \Rightarrow y_t = 2 - \frac{\varepsilon}{3}$

$$\Lambda_u = \ell^{-\varepsilon} \Rightarrow y_u = -\varepsilon$$

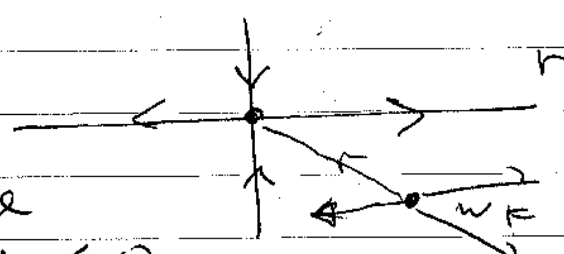
RG flow

$d < 4$ $\qquad \varepsilon = 4 - d$

$$2 - \frac{\varepsilon}{3} = 2 - \frac{4-d}{3} = \frac{2}{3} + \frac{d}{3}$$



$d > 4$



unphysical because $u < 0$

critical exponents

$$\nu = \frac{1}{y_t} = \frac{1}{2}\left(1 + \frac{\varepsilon}{6}\right)$$

$$\frac{1}{\delta} = \frac{d - y_h}{y_h}$$

$k = 0$ (nothing to be integrated)

$$h S_k' = h z S_k$$

$$\Rightarrow h' = h z = h\, \ell^{1+\frac{d}{2}}$$

$$\Rightarrow y_h = 1 + \frac{d}{2}$$

$$\Rightarrow \frac{1}{\delta} = \frac{d - 1 - \frac{d}{2}}{1 + \frac{d}{2}} = \frac{d-2}{d+2} = \frac{d-4+2}{d-4+6} = \frac{2-\varepsilon}{6-\varepsilon}$$

$$= \frac{1}{3}\left(\frac{1 - \frac{\varepsilon}{2}}{1 - \frac{\varepsilon}{6}}\right) = \frac{1}{3}\left(1 - \frac{\varepsilon}{3} + \cdots\right)$$

$$\Rightarrow \delta = 3\left(1 + \frac{\varepsilon}{3}\right) = 3 + \varepsilon$$

$$\alpha = 2 - \frac{d}{y_t} = \frac{\varepsilon}{6}$$

$$\beta = \frac{d - y_h}{y_t} = \frac{1}{2} - \frac{\varepsilon}{6}$$

$$\gamma = \frac{2y_h - d}{y_t} = 1 + \frac{\varepsilon}{6}$$

$$\eta = 2 + d - 2y_h = 0$$

Our model has the Z_2 symmetry of the Ising model. Let us compare the critical exponents to the critical exponents of the Ising model

	ε-expansion	3d Ising
α	0.167	0.11
β	0.333	0.33
γ	1.167	1.24
δ	4	4.8
ν	0.583	0.63
η	0	0.03

Pretty Good!