22b Calculation of $\langle V \rangle_0$

$$\int dk_i \, \hat{S}_{k_i} = \int_0^{\Lambda/e} dk_i \, \hat{S}_e'(k_i) + \int_{\Lambda/e}^{\Lambda} d^d k_i \, \hat{\sigma}_e'(k_i)$$

in total we obtain 16 terms from the 4 integrals. Exploiting the symmetries of the integrand we get

$$V = \tfrac{1}{4} u_0 \int_{\Lambda/e}^{\Lambda} dk_1 \ldots \int_{\Lambda/e}^{\Lambda} dk_4 \, (2\pi)^d \, \delta(k_1 + \ldots + k_4) \, \hat{\sigma}_e(k_1) \ldots \hat{\sigma}_e(k_4)$$

$$+ \tfrac{1}{4} u_0 \, 4 \int_{\Lambda/e}^{\Lambda} \int_{\Lambda/e}^{\Lambda} \int_{\Lambda/e}^{\Lambda} \int_0^{\Lambda/e} (2\pi)^d \, \delta(k_1 + \ldots + k_4) \, \hat{\sigma}_e(k_1) \ldots \hat{\sigma}_e(k_3) \, \hat{S}_e'(k_4)$$

$$+ \tfrac{1}{4} u_0 \, 6 \int_{\Lambda/e}^{\Lambda} \int_{\Lambda/e}^{\Lambda} \int_0^{\Lambda/e} \int_0^{\Lambda/e} (2\pi)^d \, \delta(k_1 + \ldots + k_4) \, \hat{\sigma}_e(k_1) \hat{\sigma}_e(k_2) \hat{S}_e'(k_3) \hat{S}_e'(k_4)$$

$$+ \tfrac{1}{4} u_0 \, 4 \int_{\Lambda/e}^{\Lambda} \int_0^{\Lambda/e} \int_0^{\Lambda/e} \int_0^{\Lambda/e} (2\pi)^d \, \delta(k_1 + \ldots + k_4) \, \hat{\sigma}_e(k_1) \cdot \hat{S}_e'(k_2) \cdots \hat{S}_e'(k_4)$$

$$+ \tfrac{1}{4} u_0 \int_0^{\Lambda/e} \ldots \int_0^{\Lambda/e} (2\pi)^d \, \delta(k_1 + \ldots + k_4) \, \hat{S}_e'(k_1) \ldots \hat{S}_e(k_4)$$

each of these terms has to be averaged w.r.t. H_{σ_e}

H_σ is a quadratic function of σ

$$\Rightarrow \langle \hat{\sigma}_k \rangle = 0 \qquad \langle \hat{\sigma}_{k_1} \hat{\sigma}_{k_2} \hat{\sigma}_{k_3} \rangle = 0$$

$$\text{eg} \int x^3 e^{-x^2} = 0$$

The σ-integration

We only have to do the integrals

$$\langle \sigma\sigma \rangle \qquad \text{and} \qquad \langle \sigma\sigma\sigma\sigma \rangle$$

↑ this just contributes as an overall constant which does not affect the effective action
(It does not depend on the S_k)

$$H_\sigma = \frac{1}{2}\int_{\Lambda/l}^{\Lambda} dk \, (r_0+k^2)\, |\hat{\sigma}_l(k)|^2$$

$$\Rightarrow \langle \hat{\sigma}_l(k_1)\,\hat{\sigma}_l(k_2)\rangle = \frac{1}{2}\int \hat{\sigma}_l(k_1)\,\hat{\sigma}_l(k_2)\, e^{-\frac{1}{2}\sum_{\Lambda/l<|k|<\Lambda}(r_0+k^2)\,\hat{\sigma}_l(k)\,\hat{\sigma}_l^*(k)} \prod_k d\sigma_l(k)$$

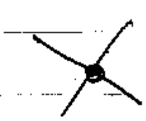
$$= V\,\frac{\delta_{k_1+k_2,0}}{r_0+k_1^2} = \frac{(2\pi)^d\,\delta(k_1+k_2)}{r_0+k^2}$$

note that $\hat{\sigma}_l(-k) = \hat{\sigma}_l^*(k)$

22c) Results from one loop renormalization

Only two terms

(Each integral is d-dimensional).



$$\langle V\rangle_0^{ssss} = \frac{u_0}{4}\int_0^{\Lambda/l} dk_1 \ldots dk_4\,(2\pi)^d\,\delta^d(k_1+\ldots+k_4) \times \hat{s}_l'(k_1)\ldots\hat{s}_l'(k_4)$$

$$k_l = l\,k \qquad \hat{s}_l(k) = \frac{1}{z}\,\hat{s}_l'(k) \Rightarrow u_{4l} = u_0 z^4 \frac{1}{l^{4d}}\, l^d = u_0 z^4 l^{-3d}$$

$$\langle V \rangle_0^{\sigma\sigma SS} = \frac{6}{4} u_0 \int_0^{\Lambda/e} d^d k_3\, d^d k_4\, (2\pi)^d\, \delta^d(k_3+k_4)$$

$$(2\pi)^d \int_{\Lambda/e}^{\Lambda} \frac{1}{r_0+k_1^2} d^d k_1 \quad S_e'(k_3)\, S_e'(k_4)$$

$$\delta^d(k_1+k_2+k_3+k_4)\, \delta^d(k_1+k_2) = \delta^d(k_1+k_2)\, \delta^d(k_3+k_4)$$

after rescaling

$$\langle V \rangle_0^{\sigma\sigma SS} = \frac{6}{4} u_0 \frac{z^2}{e^{2d}} e^d \int_0^{\Lambda} dk_e |S_e(k_e)|^2 (2\pi)^d \int_{\Lambda/e}^{\Lambda} dk \frac{1}{r_0+k^2}$$

From the quadratic part of the Hamiltonian we get

$$\frac{1}{2} \int_{|k|<\Lambda/e} dk\, (r_0+k^2)\, |\hat{S}_e'(k)|^2$$

$$= \frac{1}{2} \frac{z^2 e^{-d}}{e^2} \int_{|k|<\Lambda} dk_e\, (\underbrace{r_e}_{e^2 r_0} + k_e^2)\, |\hat{S}_e(k)|^2$$

Total result for u_{2e}

$$u_{2e} = z^2 e^{-d} (r_0 + \frac{k_e^2}{e^2} + \frac{6}{2} u_0 (2\pi)^d \int_{\Lambda/e}^{\Lambda} \frac{d^d k}{r_0+k^2}$$

choose z such that coeff of k_e^2 is 1

$\Rightarrow z^2 e^{2-d} = 1$

$$\Rightarrow u_{2e} = (r_0 e^2 + k_e^2 + 3 u_0 e^2 \int_{\Lambda/e}^{\Lambda} \frac{d^d k}{r_0+k^2}$$

$$u_{4e} = u_0\, e^{4-d}$$

u_4 renormalizes to a trivial fixed point for $d > 4$. ($u_{4\ell} \to 0$ for $\ell \to \infty$)

for $d < 4$, $u_{4\ell} \to \infty$ for $\ell \to \infty$

$\nexists$ no nontrivial fixed point

We have to do better!

(22a) Calculation of $\langle V^2 \rangle_0 - \langle V \rangle_0^2$

To do this efficiently we have to introduce Feynman rules

vertex $\times$ $\frac{u_0}{4} (2\pi)^d \, (k_1 + \dots + k_4)$

propagator $\overline{k_1 \qquad k_2}$ $(2\pi)^d \, \frac{\delta^d(k_1 + k_2)}{r_0 + k_1^2}$

A Gaussian integral is equal to the sum of all pairwise contractions

$\langle V^2 \rangle_0 - \underbrace{\langle V \rangle_0 \langle V \rangle_0}$

no contraction between these two factor because they are separate integrals. They correspond to disconnected diagrams

$\Rightarrow \langle V^2 \rangle_0 - \langle V \rangle_0^2$ contains only connected diagrams

A vertex can have s and σ legs. The σ legs have to be connected by a propagator. The s-legs are external

Notation $- - - - \sigma$
$\overline{\qquad\quad} s$

Vertices