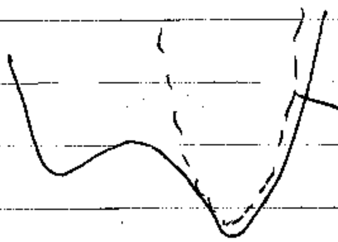
2 b) Basic idea

$$H^{eff}(s) = \int d^d r \left(\tfrac{1}{2}(\vec{\nabla} s)^2 + \tfrac{1}{2} r_0 s^2 + \tfrac{1}{4} u_0 s^4 - h_0 s \right)$$

Mean field theory works for $d > 4$. Then fluctuation about the minimum are small.



$V(s)$

$$V(s) = \tfrac{1}{2} r_0 s^2 + \tfrac{1}{4} u_0 s^4 - h_0 s$$

Gaussian approximation is good

Gaussian fixed point $u_0^* = h_0^* = r_0^* = 0$

in dimensionless variables
$$\bar{u}_0 = u_0 \, r_0^{\frac{d-4}{2}}$$
$$\bar{h}_0 = h_0 \, r_0^{-(1+\frac{d}{2})/2}$$

u_0 is irrelevant
h_0 is relevant   For d > 4

For $d < 4$ the Gaussian fixed point becomes unstable and a new fixed point appears. This is the Wilson–Fisher fixed point.

How come?

$$[u_0] = L^{d-4} \qquad u_0' = \frac{u_0}{\ell^{d-4}} = \ell^{\varepsilon} u_0$$

$$\frac{\partial u}{\partial \ell} = \varepsilon \ell^{\varepsilon - 1} u_0 = \frac{\varepsilon}{\ell} u$$

$$\frac{\partial u}{\partial s} = \varepsilon u \quad \text{with } s = \log \ell$$

including next order we find

$$\frac{\partial u}{\partial s} = \varepsilon u - A u^2$$

two fixed points $u^* = 0$

$$u^* = \frac{\varepsilon}{A}$$

2(c) Gaussian model

$$H(s) = \int d^d r \left(\frac{1}{2}(\nabla s)^2 + \frac{1}{2} r_0 s^2 \right.$$

momentum space representation

$$H(s) = \frac{1}{V} \int_{0<|K|<\Lambda} \frac{d^d K}{(2\pi)^d} \frac{1}{2} |\hat{S}_K|^2 (r_0 + K^2)$$

$$S(r) = \frac{1}{V} \sum_K e^{iKr} \hat{S}_K \qquad dn = \frac{L}{2\pi} dK$$

$S(r)$ is real $\Rightarrow$ $\hat{S}_K = \hat{S}^*_{-K}$

RG transformation: • integrate out momenta
• rescale space

block spins $\hat{S}_e'(K) = \hat{S}_K$, $0<|\vec{K}|<\frac{\Lambda}{e}$
$\sigma_e'(K) = \hat{S}_K$, $\frac{\Lambda}{e}<|\vec{K}| \leq \Lambda$
to be integrated out
(short distance d.o.f.)

$$H(\hat{S}_K) = H(S_K') + H(\hat{\sigma}_{e'}(K))$$

Because we have a Gaussian model things are simple

$$Z = \int DS\, e^H = \int_{0<|\vec{K}|<\Lambda} \prod dS_K\, e^{H(\hat{S}_K)}$$

$$= \int_{0<|\vec{K}|<\frac{\Lambda}{e}} \prod_K dS_e'(K)\, e^{H(S_e'(K))} \prod_K d\sigma_e'(K)\, e^{H(\sigma_e'(K))}$$

$$= Z_s Z_\sigma \quad , \quad Z_\sigma = \prod_{\Lambda/\ell < \vec{k} < \Lambda} \left(\frac{2\pi \dot{V}}{r_0^2 + k^2} \right)$$

$$Z_s = \int D S_\ell' \; e^{-\frac{1}{V}\int_0^{\Lambda/\ell} \frac{d^d k}{(2\pi)^d} \frac{1}{2}(r_0 + k^2)\, |S_\ell'(k)|^2}$$

rescale integration variables

$$k_\ell = \ell k$$
$$r_\ell = r_0' = r_0 \ell^2$$
$$\hat{S}_\ell' = \ell^{d/2+1} \hat{S}_\ell$$

Then

$$Z_s = \int D \hat{S}_\ell \; e^{-\frac{1}{2}\int_0^{\Lambda} \frac{d^d k_\ell}{(2\pi)^d} (r_\ell + k_\ell^2)\, |\hat{S}_\ell(k)|^2}$$

RG recursion relation; $\boxed{r_\ell = r_0 \ell^2}$

fixed point $\quad r^* = 0$

$$\Lambda_\ell = \ell^2 \Rightarrow y_r = 2$$

$$r_0 \sim t \Rightarrow y_r = y_t \Rightarrow \nu = \frac{1}{y_t} = \frac{1}{2}$$

this is the Mean Field value

infinitesimal RG equation

$$r_{\ell+\delta\ell} = (\ell^2 + 2\ell\,\delta\ell)\, r_0 = r_\ell + \frac{2 r_\ell}{\ell} \delta\ell$$

$$\Rightarrow \frac{\partial r_\ell}{\partial \ell} = \frac{2 r_\ell}{\ell} \Rightarrow \frac{\partial r_s}{\partial s} = 2 r_s$$

$$s = \log \ell$$

2(a) RG eq with magnetic field

Hamiltonian with magnetic field

$$H \to H + \underbrace{h_0 \int d^d r \, S(r)}_{h_0 \hat{S}_0} \qquad S(r) = \frac{1}{V} \sum_K e^{iKr} \hat{S}_K$$

$\Rightarrow$ only the zero momentum mode contributes
nothing to be integrated out

$$h_0 \hat{S}_0 = h_0 \hat{S}'_\ell(0)$$

$$\hat{S}'_\ell(0) = \ell^{d/2+1} \hat{S}_\ell(0) \Rightarrow h_\ell = h_0 \ell^{\frac{d}{2}+1}$$

$$\Rightarrow y_h = \frac{d}{2} + 1$$

$$\Rightarrow \delta = \frac{y_h}{d - y_h} = \frac{\frac{d}{2}+1}{\frac{d}{2}-1}$$

$\delta \neq 3$ disagrees with MFT even for $d > 4$

How can this happen?

21 e) Dangerous irrelevant variables

$$f_s(t,h,u_0) = \ell^{-d} f_s(t\ell^{y_t}, h\ell^{y_h}, u_0\ell^{y_u})$$

magnetization $\quad M \sim \partial_h f_s \sim \ell^{y_h-d} F(t\ell^{y_t}, h\ell^{y_h}, u_0\ell^{y_u})$

We study δ ; we can't put $t=0$

$$M \sim \ell^{y_h-d} F(0, \underbrace{h\ell^{y_h}}_{\text{choose}=1}, u_0\ell^{y_u})$$

$$\Rightarrow M \sim h^{\frac{d-y_h}{y_h}} F\left(0, 1, u_0 h^{-\frac{y_u}{y_h}}\right)$$

$M \sim \phi$ with $u_0 \phi^3 = h$

MFT $\Rightarrow$ $M(0,h,u_0) \sim u_0^{-\frac{1}{3}}$

d-y_u

In MFT domain $\quad F(0,1,u_0 h^{-y_u/y_h}) \sim u_0^{-\frac{1}{3}} h^{\frac{y_u}{3y_h}}$

$$\Rightarrow M \sim h^{\frac{d-y_h}{y_h} + \frac{y_u}{3y_h}}$$

$y_h = 1 + \frac{d}{2}$ $\qquad$ naive scaling of $u_0 \sim L^{d-4}$

$\Rightarrow y_u = 4-d$

$$\Rightarrow \frac{d}{y_h} - 1 + \frac{y_u}{3y_h} = \frac{d}{1+\frac{d}{2}} - 1 + \frac{4-d}{3(1+\frac{d}{2})} =$$

$$\frac{3d-d+4}{3(1+\frac{d}{2})} - 1 = \frac{4}{3} - 1 = \frac{1}{3}$$

MFT result

22a) The full theory

$$H = \int d^d r \left(\tfrac{1}{2}(\nabla S)^2 + \tfrac{1}{2} r_0 S^2 + \tfrac{1}{4} u_0 S^4 - h_0 S \right)$$

Fourier transform of S^4 term

$$\int d^d r\, S^4 = \int d^d r \frac{1}{V} \sum_{k_1 \cdots k_4} e^{i(k_1 + \cdots + k_4) r} \hat{S}_{k_1} \cdots \hat{S}_{k_4}$$

$$= \frac{1}{V^3} \sum_{k_1 \cdots k_4} \delta^d (k_1 + \cdots + k_4)\, \hat{S}_{k_1} \cdots \hat{S}_{k_4}$$

$$k_i = \frac{2\pi n_i}{L} \qquad \hat{d} n_i = \frac{d^d k_i}{(2\pi)^d} L^d$$

$$\delta^{\wedge}_{Kron}(k_1 + \cdots + k_4) = \left(\frac{2\pi}{L}\right)^{d} \delta^d_{Dirac}(k_1 + \cdots + k_4)$$

$$\Rightarrow \int d^d r\, S^4(r) = \int \hat{d} k_1 \cdots d k_4\, \hat{S}_{k_1} \cdots \hat{S}_{k_4} (2\pi)^d \delta(k_1 + \cdots + k_4)$$

We integrate out the high momentum variables

$$\hat{\sigma}_\ell'(k) = \hat{S}_\ell(k) \qquad \frac{\Lambda}{\ell} < |\vec{k}| < \Lambda$$

$$\hat{S}_\ell'(k) = \hat{S}_\ell(k) \qquad |\vec{k}| < \frac{\Lambda}{\ell}$$

$$H(S) = H_{S'}(\hat{S}_\ell') + H_{\sigma'}(\hat{\sigma}_\ell') + \underset{\uparrow \text{quartic}}{V}(S_\ell', \sigma_\ell')$$

aim

renormalized Hamiltonian

$$H_\ell = \tfrac{1}{2} \int_0^{\Lambda} \hat{d} k \;\; u_{2\ell} |\hat{S}_k|^2$$

$$+ \tfrac{1}{4} a_{4,\ell} \int_0^{\Lambda} \hat{d}^d k_1 \cdots d k_4\, (2\pi)^d \delta(k_1 + \cdots + k_4) \hat{S}_{k_1} \cdots \hat{S}_{k_4}$$

We first consider $h_0 = 0$

$$Z(r_0, u_0) = \int \mathcal{D}S e^{-H} = \int \mathcal{D}S_e' \, e^{-H_{S_e'}(S_e')} \int \mathcal{D}\sigma_e' \, e^{-H_{\sigma_e'}(\sigma_e')} \times e^{-V(\sigma_{e'}, S_{e'})}$$

We divide the partition function by

$$\int \mathcal{D}\sigma_{e'} \, e^{-H_{\sigma_{e'}}(\sigma_{e'})}$$

This is a constant that does not affect the effective action for S_e'

$$\Rightarrow Z(r_0, u_0) = \int \mathcal{D}S_{e'} \, e^{-H_{S_e'}} \langle e^{-V} \rangle_0$$

As in the Ising model, we use a cumulant expansion

$$\langle e^{V} \rangle_0 = e^{\langle V \rangle_0 + \frac{1}{2}\langle V^2 \rangle_0 - \frac{1}{2}\langle V \rangle_0^2}$$

In order to get the new Hamiltonian, we have to rescale the spins and the momenta

$$S_e' = z \hat{S}_e(\kappa) \qquad K_e = e K$$

Then

$$H_e(\hat{S}_e(\kappa_e)) = H_{S'}(z \hat{S}_e(\kappa_e)) + \langle V \rangle_0 + \frac{1}{2}\langle V^2 \rangle_0 - \frac{1}{2}\langle V \rangle_0^2$$

$$= \frac{1}{2}\int u_{2e} \, |\hat{S}_e(\kappa)|^2 + \frac{1}{2} u_{4e} \int (2\pi)^d \, \delta^d(\kappa_1 + \cdots + \kappa_4) \, \hat{S}_e(\kappa_1) \cdots \hat{S}_e(\kappa_4) + O(\hat{S}_e^6(\kappa))$$

We now have to calculate u_{2l} and u_{4l}