Divide H into two pieces

$$H = H_0 + V \qquad H_0 = K \sum_I \sum_{ij \in I} S_i S_j$$

$$V = K \sum_{I \neq J} \sum_{\substack{i \in I \\ j \in J}} S_i S_j$$

average with respect to H_0

$$\langle A \rangle_0 = \frac{\sum_{\sigma_I} e^{H_0(S_I, \sigma_I)} A(S_I, \sigma_I)}{\sum_{\sigma_I} e^{H_0(S_I, \sigma_I)}}$$

$$e^{H'(S_I)} = \langle e^V \rangle_0 \underbrace{\sum_{\sigma_I} e^{H_0(S_I, \sigma_I)}}_{Z_0^M}$$

M = # of blocks

$$Z_0(K) = \sum_{S_1 S_2 S_3} e^{K(S_1^I S_2^I + S_1^I S_3^I + S_2^I S_3^I)}$$

$$= \delta_{S_I 1}\left(e^{3K} + 3e^{-K}\right) + \delta_{S_I -1}\left(e^{3K} + 3e^{-K}\right)$$

$$= e^{3K} + 3e^{-K}$$

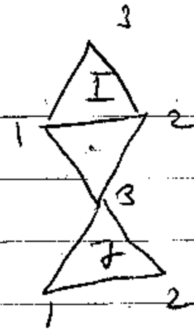
We calculate $\langle e^V \rangle$ by means of the cumulant expansion

$$\langle e^V \rangle = e^{\langle V \rangle_0 + \frac{1}{2}\langle V^2 \rangle - \frac{1}{2}\langle V \rangle_0^2 + \dots}$$

exact for gaussian distribution

$$V = \sum_{I \neq J} V_{IJ}$$

sum over links



$$= K\, S_{3J}\,(S_{1I} + S_{2I})$$

$$\Rightarrow \langle V \rangle_0 = 2K \langle S_{3J}\, S_{1I} \rangle_0 = 2K \langle S_J^3 \rangle_0 \langle S_I^1 \rangle_0$$

H_0 does not couple different blocks

$$\langle S_J^3 \rangle_0 = \frac{1}{Z_0} \sum_{\sigma_J} S_3\, e^{K(S_1^J S_2^J + S_1^J S_3^J + S_2^J S_3^J)}$$

$$= \frac{1}{Z_0}\, \delta_{S_J,1} \left(e^{3K} + e^{-K}\right) + \frac{1}{Z_0}\, \delta_{S_J,-1} \left(-e^{3K} - e^{-K}\right)$$

\delta_{ S_J,1}

$$= \frac{S_J}{Z_0} \left(e^{3K} + e^{-K}\right)$$

$$\langle S_I^1 \rangle_0 = \frac{S_I}{Z_0} \left(e^{3K} + e^{-K}\right)$$ same argument

$$\Rightarrow H(\{S_I\}) = M \log Z_0(K) + \langle V \rangle_0$$

$$= M \log Z_0(K) + K' \sum_{\langle IJ \rangle} S_I S_J$$

$$K' = 2K\, \phi(K)^2 \qquad \phi(K) = \frac{e^{3K} + e^{-K}}{e^{3K} + 3e^{-K}}$$

(12)

20 b) Analysis of RG equation

Fixed point $K^* = 2K^* \phi^2(K^*)$

$$\Rightarrow K^* = 0 \text{ or } \frac{e^{3K^*} + e^{-K^*}}{e^{3K^*} + 3e^{-K^*}} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow e^{4K^*} + 1 = \frac{1}{\sqrt{2}}\left(e^{4K^*} + 3\right)$$

$$\Rightarrow e^{4K^*}\left(1 - \frac{1}{\sqrt{2}}\right) = \left(\frac{3}{\sqrt{2}} - 1\right)$$

$$\Rightarrow e^{4K^*} = \frac{3 - \sqrt{2}}{\sqrt{2} - 1} = 1 + 2\sqrt{2} \Rightarrow K^* = 0.34$$

Onsager $K^* = 0.~~xxx~~$ 0.44 but this is for a square lattice. For a triangular lattice it may be otherwise (K^* = 0.274, but critical exponents are the same)

Linearize RG Flow

$$\Lambda_t = \frac{\partial K'}{\partial K}\Big|_{K^*} = 2\phi(K^*) + 2K^*\phi'(K^*)$$
$$= 1.62$$

$$\Lambda_t = \ell^{y_t} = (\sqrt{3})^{y_t} \Rightarrow y_t = \frac{\log 1.62}{\log 1.73} = 0.88$$

$$\Rightarrow \alpha = 2 - \frac{d}{y_t} = -0.27$$

Onsager $\Rightarrow \alpha = 0$ for 2d Ising

20c) <u>With magnetic field</u>

Near T_c ; $H \to 0$ so we include H in V

Additional contribution

$$\langle H \sum_I \sum_{i \in I} S_i \rangle_0$$

$$= \frac{1}{Z_0} H \sum_I \delta_{\underline{S}_I 1} \left(3e^{3K} + 3e^{-K}\right)$$

$$+ \delta_{\underline{S}_I -1}\left(-3e^{-K} - 3e^{-K}\right)$$

$$= 3 H S_I \phi(K) \Rightarrow H' = 3H\phi(K)$$

$$\left.\frac{\partial H'}{\partial H}\right|_{\substack{K=K^* \\ H=0}} = 3\phi(K^*) = 2.12 = \lambda^h$$

$$\lambda^h = (\sqrt{3})^{y_h} \Rightarrow y_h = 1.37$$

$$\delta = \frac{y_h}{d - y_h} = \frac{1.37}{2 - 1.37} = 2.17$$

Onsager value : $\delta = 15$.

We conclude that we have to improve the RG group.

20 a) <u>Phase diagram</u>

$$H' = 3\,\phi(K)\,H \qquad \phi(K) = \frac{e^{3K} + e^{-K}}{e^{3K} + 3e^{-K}}$$
$$K' = 2K\,\phi^2(K)$$

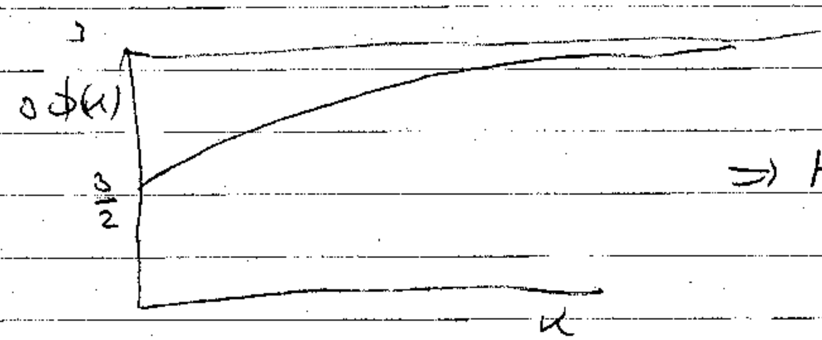
Fixed points $\quad H = 0,\ \infty,\ -\infty$
$\quad K = 0,\ K_c,\ \infty$

for $K \to 0 \quad \phi(K) \simeq \dfrac{1+3K-K+1}{1+3K+3(1-K)} = \dfrac{2+2K}{4+2K} = \frac{1}{2} + O(K)$

$\Rightarrow K' \simeq \frac{1}{2} K$

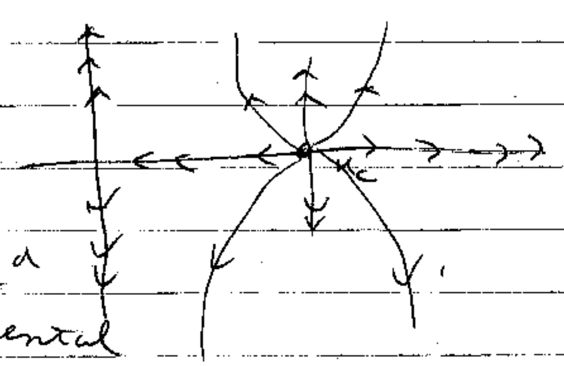
for $K \to \infty \quad K' = 2K$

$\Rightarrow K = 0$ and $K = \infty$ are attractive fixed points



$\Rightarrow H = \infty$ is attractive

phase diagram



for $K = \infty$
$\lambda_h = 3 = (\sqrt{3})^2 = e^d$
This is <u>not</u> accidental

20e) Wienhuis-Nauenberg criterium

Application of RG group to first order transition

magnetization $\quad M(K^{(0)}) = \frac{\partial g}{\partial h}\Big|_{h=0^+}$

$$g(K) = \frac{1}{N} \log Z_N(K)$$

$$\Rightarrow \quad \partial_h \, \ell^{-d} g(K') = \frac{\partial h'}{\partial h} \ell^{-d} \partial_{h'} g(K')$$

$$= \underbrace{\frac{\partial h'}{\partial h}}_{\alpha(K^{(0)})} \ell^{-d} M(K')$$

$$\Rightarrow M(K^{(0)}) = \frac{\alpha(K^{(0)})}{\ell^d} M(K') = \frac{\alpha(K^{(0)})}{\ell^d} \frac{\alpha(K^{(1)})}{\ell^d} \cdots \frac{\alpha(K^{(n)})}{\ell^d} M(K^{(n+1)})$$

$$\underset{K_c}{\times} \longrightarrow \underset{T=0}{K=\infty}$$

For a first order transition we have that

$$M(T, 0^+) - M(T, 0^-) \neq 0$$

$$\Rightarrow M(K^{(i)}) \neq 0 \quad \forall i$$

$$\Rightarrow \lim_{n\to\infty} \frac{\alpha(K^{(n)})}{\ell^d} = 1$$

for $n \to \infty$ we are at the $T=0$ fixed point

$$\Rightarrow \frac{\partial h'}{\partial h} = \lim_{n\to\infty} \alpha(K^{(n)}) = \ell^d \Rightarrow \lambda^h = \ell^d \text{ at } T^* = 0$$

20 f) RG for correlation functions

$$G(r-r', t, h) = \frac{1}{\ell^{2d}} \frac{\partial^2 \log Z(t,h)}{\partial h(r)\, \partial h(r')}$$

$$= \frac{1}{\ell^{2d}} \frac{\partial^2 \log Z(t', h')}{\partial h'(r)\, \partial h'(r')} \frac{\partial h'(r)}{\partial h(r)} \frac{\partial h'(r')}{\partial h(r')}$$

$$= G\left(\frac{r-r'}{\ell}, t', h'\right) \cdot \frac{\Lambda_h^2}{\ell^{2d}}$$

extra factor ℓ^d because we get a factor ℓ^d less spins when differentiating with respect to $h'(r)$ or $h'(r')$

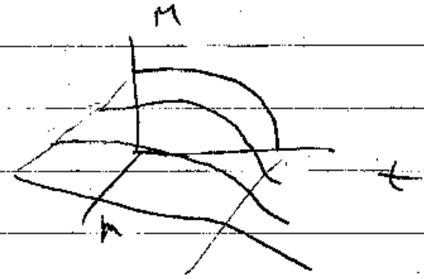
20 g) Crossover transition

If $h \neq 0$ we do not have a sharp phase transition but a smooth crossover.

$f_s(t, h \neq 0)$ is a smooth function of $t \Rightarrow$

$M \sim \partial_h f_s(t, h \neq 0)$ is a smooth function of t

Experimentally always $h \neq 0$



if $\frac{h^{1/\delta}}{|t|^\beta} \gg 1$ we see δ

$\frac{h^{1/\delta}}{|t|^\beta} \ll 1$ we see β

$h \to 0 \quad M \sim t^\beta$

$t \to 0 \quad M \sim h^{1/\delta}$

20h) Finite size scaling

problem no exact phase transitions do occur in finite systems. Hence, we wish to study phase transitions by means of Monte Carlo simulations

For a system with volume L^d we have

$$f_s(t, h, L^{-1}) = \ell^{-d} f_s(t', h', \frac{1}{(L/\ell)})$$

$$= \ell^{-d} f_s(t\ell^{y_t}, h\ell^{y_h}, \frac{\ell}{L})$$

$\Rightarrow$ $\frac{1}{L}$ appears as a relevant coupling constant

$$\Lambda_L = \ell \qquad y_L = 1$$

phase transition is at $t = t^*$, $h^* = 0$, $\frac{1}{L^*} = 0$

For $\frac{1}{L} \neq 0$ we have a smooth crossover

$f_s(t, h, \frac{1}{L} \neq 0)$ is a smooth function of t and h

Example correlation length at $h = 0$

$$\xi(t, \tfrac{1}{L}) = \ell\, \xi(t^{\,y_t}, \ell \tfrac{1}{L})$$

choose $\ell^{y_t} t = 1 \Rightarrow$ $= t^{-\nu}\,\xi(1, t^{-\nu}\frac{1}{L})$

$= t^{-\nu} F(t^{-\nu}\frac{1}{L})$

$$= t^{-\nu} L t^{\nu} \bar{F}(L t^{\nu})$$
$$= L \bar{F}(L t^{\nu})$$

for $L \to \infty$ at fixed $t \ll 1$

$$\xi(t, \tfrac{1}{L}) \sim t^{-\nu} \Rightarrow \bar{F}(x) \sim \frac{1}{x}$$

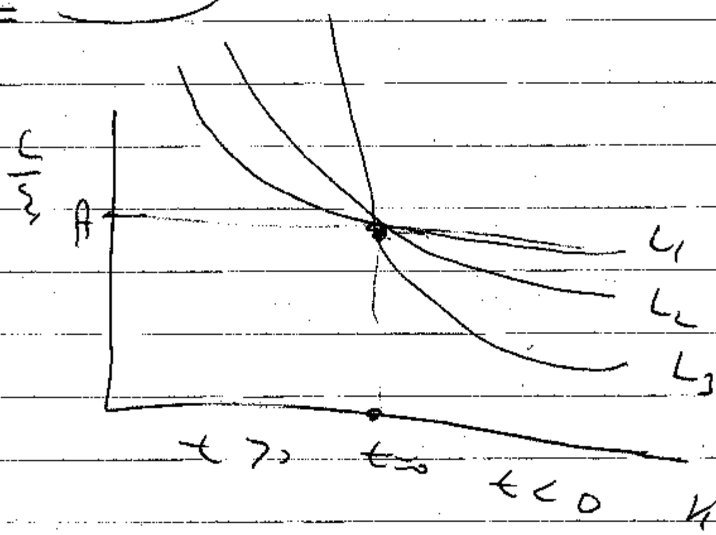
For L finite and $t \to 0$ the correlation length is $\mathcal{O}(L)$; larger values are not possible

$\Rightarrow \bar{F}(0) =$ constant

At finite L $\quad \frac{L}{\xi} = A + B t L^{1/\nu}$

$\bar{F}$ is a regular function of t

$\frac{L}{\xi} = \frac{1}{\bar{F}} =$



$\frac{1}{\nu}$ can be obtained from L dependence

2(a) Epsilon expansion

$$H_{eff} = \int d^d r \left(\tfrac{f}{2}(\nabla\phi)^2 + \tfrac{1}{2} r_0 \phi^2 + \tfrac{1}{4} u_0 \phi^4 \right)$$

$$[\phi] = L^{1-\frac{d}{2}}, \quad r_0 = \frac{1}{L^2}, \quad u_0 = L^{d-4}$$

$$Z = \int D\phi \, e^{-H_{eff}}$$

length scale $\frac{1}{\sqrt{r_0}} \Rightarrow u_0' = u_0 r_0^{\frac{d-4}{2}}$

$r_0 \sim t \Rightarrow u_0'$ diverges for $t \to 0$ and $d < 4$

in momentum space this is a divergence for $k \to 0$ (infrared divergence)

How to deal with this divergence?

Momentum shell RG; construct RG recursion relation by integration over momenta $\frac{\Lambda}{\ell} < k < \Lambda$

fixed points with $u_0^* \neq 0$ can be accessed by an expansion in $\varepsilon = 4-d$