19a Fixed points

Renormalization group transformation
$R_\ell(K)$

fixed point $R_\ell(K^*) = K^*$

but correlation length $\xi' = \xi/\ell$
but also $K' = K^* \Longrightarrow \xi' = \xi$

this is only possible if $\xi = \infty$, critical fixed point
$\xi = 0$, trivial fixed point

Theorem: all points in the basin of attraction of a critical fixed point have infinite correlation length

$K_0 \to K_1 \to K_2 \to K_3 \qquad \to K_N \to \to K^*$
$\xi \quad \frac{\xi}{\ell} \quad \frac{\xi}{\ell^2} \qquad\qquad N \to \infty$

For $N \to \infty$ $\xi(K_N) = \infty$

$$\xi(K_N) = \frac{\xi(K)}{\ell^N} \Rightarrow \xi(K) = \infty$$
because $\ell > 1$

critical manifold basin of attraction of a critical point

will be classified according to its codimension. For example, if all points of the parameter space flow into the fixed point the codimension is zero

(19e) Linearized RG flow

We expand the RG flow around the fixed point K_n^*

$$K_n = K_n^* + \delta K_n \qquad K_n' = K_n^* + \delta K_n'$$

$$K_n' = R_{\ell^n}(K_n^* + \delta K_n)$$

$$= K_n^* + \frac{\partial R_{\ell^n}}{\partial K_m}\delta K_m + \cdots$$

$$\text{or} \quad \delta K_n' = \sum_m M_{nm}\,\delta K_m$$

Generally, M is not a symmetric matrix, then the left and right eigenvector are different and the eigenvalues may be complex.

We consider the case of a real symmetric matrix M_{mn}

eigenvalue equation $M e_\mu = \lambda_\mu e_\mu$

semigroup property $M_{\ell_2} M_{\ell_1} = M_{\ell_1} M_{\ell_2} = M_{\ell_1 \ell_2}$
$\forall \ell_1, \ell_2$

$\Rightarrow [M_{\ell_1}, M_{\ell_2}] = 0 \Rightarrow$ we can find a common set of eigenvectors

For the eigenvalues we find $\Lambda_{\ell_1} \Lambda_{\ell_2} = \Lambda_{\ell_1 \ell_2}$

Consider a function with the property

$$f(x)\, f(y) = f(xy)$$

then $\quad f(x)\, f'(y)\big|_{y=1} = x f'(x y)\big|_{y=1}$

$$\Rightarrow f(x)\, f'(1) = x f'(x)$$

$$\Rightarrow \quad d \log f(x) = f'(1)\, d \log x$$

$$\Rightarrow f(x) = c\, x^{f'(1)}$$

$\Rightarrow \lambda_\ell = \ell^{y_\sigma}$ with y_σ an ℓ-independent constant

Behavior under a RG transformation

$K = K^* + \delta K$
$K' = K^* + \delta K'$ $\qquad \delta K' = M \delta K$

expansion in eigenvectors $\qquad \delta K' = \sum c_n' e_n$
$\qquad\qquad\qquad\qquad \delta K = \sum c_n e_n$

$$\sum c_n' e_n = \sum_n \lambda_n c_n e_n$$

$$\Rightarrow c_n' = \lambda_n c_n$$

three classes of operators

i) relevant operators $|\lambda_n| > 1$
ii) irrelevant operators $|\lambda_n| < 1$
iii) Marginal operators $|\lambda_n| = 1$

relevant directions: flow away from the fixed point

irrelevant directions: flow toward the fixed point

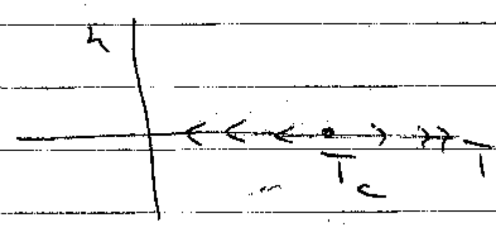
19f Classification of fixed points

i) $\xi = 0$ trivial fixed points
ii) $\xi = \infty$ critical fixed points

Classification according to the co-dimension of the basin of attraction; i.e. the number of conditions imposed in order to flow into a fixed point

codimension 0: then $\xi = 0$; the correlation length cannot diverge for all values of the parameters; it certainly does not for $T = 0$

codimension 1, eg Ising system



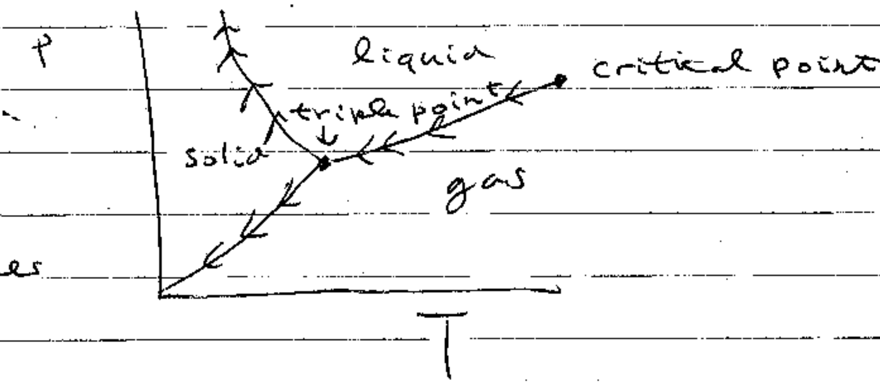
$h=0, T=0$ } fixed points with $\xi = 0$ codimension 1
$h=0, T=\infty$ } $h = 0$ is the condition

codimension 2;
$T = T_c$ } codimension-2 fixed point
$h = 0$ } $h = 0$ and $T = T_c$ have to be fixed

triple point

$\xi > 0$

intersection of two first order lines



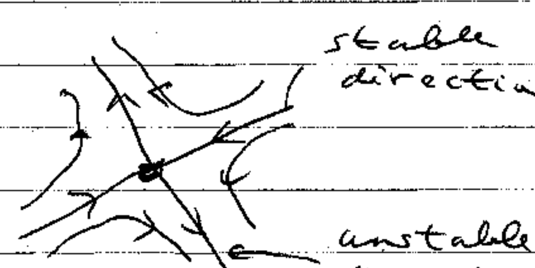
all points in one phase flow the same fixed point. If you are on a phase boundary you stay on it during renormalization. In order to flow to the triple point we have to fix two conditions

codimension > 2

$\xi = \infty$ multicritical fixed points

$\xi = 0$ multiple coexistence phases

Typical flow patter



19 g) RG transformation with one relevant variable

Then $T' = R_\ell(T)$

Fixed point $T^* = R_\ell(T^*)$

linearize

$$\delta T' = T' - T^* = R_\ell(T^* + \delta T) - T^*$$
$$= \delta T \, \partial_T R_\ell \big|_{T=T^*}$$
$$\equiv \Lambda_\ell \, \delta T$$

Semigroup $\quad \Lambda_{\ell_1} \Lambda_{\ell_2} = \Lambda_{\ell_1 \ell_2} \Rightarrow \Lambda_\ell = \ell^{y_t}$

then $\quad t' = \dfrac{\delta T'}{T^*} = \ell^{y_t} \dfrac{\delta T}{T^*} = \ell^{y_t} t$

$$\xi(t') = \frac{\xi(t)}{\ell} \Rightarrow \xi(t) = \ell \, \xi(\ell^{y_t} t)$$

For $\ell^{y_t} = t^{-1}$ $\qquad \xi(t) = t^{-\frac{1}{y_t}} \xi(1)$

$$\Rightarrow \nu = \frac{1}{y_t} = \frac{\log \ell}{\log \Lambda_\ell}$$

Free energy density : $\quad f(t) = \ell^{-d} f(t') = \ell^{-d} f(\ell^{y_t} t)$

$\quad \Rightarrow f(t) = t^{d/y_t} f(1)$ $\quad$ for $\ell^{y_t} t = 1$

Specific heat : $\quad c \sim \partial_t^2 f \sim t^{\frac{d}{y_t} - 2}$

$\sim t^{-\alpha} \Rightarrow \alpha = 2 - \dfrac{d}{y_t}$

$= 2 - \nu d$

Josephson scaling

19h Two relevant variables

Then $T' = R_\ell^T(T, H)$

$H' = R_\ell^H(T, H)$

fixed point (T^*, H^*)

linearize
$$\begin{pmatrix}\delta T' \\ \delta H'\end{pmatrix} = \underbrace{\begin{pmatrix}\partial_T R_\ell^T & \partial_H R_\ell^T \\ \partial_T R_\ell^H & \partial_H R_\ell^H\end{pmatrix}}_{M}\Bigg|_{(T^*, H^*)} \begin{pmatrix}\delta T \\ \delta H\end{pmatrix}$$

simplest case: M is diagonal $M = \begin{pmatrix}\lambda_\ell^t & \\ & \lambda_\ell^h\end{pmatrix}$

semigroup $\xi(t, h) = \ell\, \xi(\ell^{y_t} t, \ell^{y_h} h)$

$\lambda_\ell = \ell^{y_t}, \lambda_h = \ell^{y_h}$

choose $\ell^{y_t} t = 1 \Rightarrow \xi(t, h) = t^{-\frac{1}{y_t}}\, \xi\left(1, t^{-\frac{y_h}{y_t}} h\right)$

$\Rightarrow \nu = -\frac{1}{y_t}$

magnetization $m \sim \partial_h f$

$$f(t, h) = \ell^{-d} f(\ell^{y_t} t, \ell^{y_h} h)$$
$$= t^{d/y_t} f(1, t^{-y_h/y_t} h)$$

$$\Rightarrow m \sim t^{\frac{d}{y_t} - \frac{y_h}{y_t}} f'(1, t^{-y_h/y_t} h)$$

For $h = 0$ $\quad m \sim t^{\frac{d - y_h}{y_t}} \Rightarrow \beta = \frac{d - y_h}{y_t}$

$t > 0$ and $h \neq 0$ then choose $\ell^{y_h} h = 1$

$$\Rightarrow f(t,h) = h^{\frac{d}{y_h}} f\left(h^{-\frac{y_t}{y_h}} t, 1\right)$$

$$\Rightarrow m = \partial_h f \sim h^{\frac{d}{y_h} - 1} \Rightarrow \delta = \frac{y_h}{d - y_h}$$

19i) Irrelevant variables

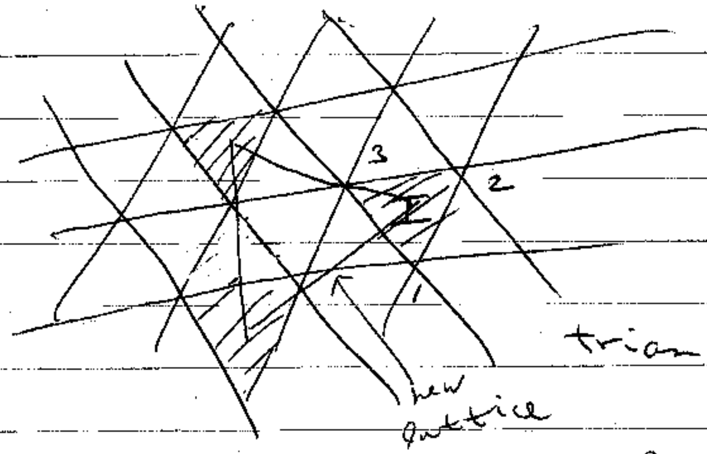
$$f(t,h,\kappa) = \frac{1}{\ell^d} f(t', h', \kappa')$$

$$= \frac{1}{\ell^d} f(\Lambda_t t, \Lambda_h h, \Lambda_\kappa \kappa)$$

$$= \frac{1}{\ell^{nd}} f(\Lambda_t^n t, \Lambda_h^n h, \underbrace{\Lambda_\kappa^n \kappa}_{\substack{\to 0 \text{ for } \Lambda_\kappa < 1 \\ \text{irrelevant}}})$$

dangerous irrelevant variables: $f(t,h,\kappa)$ is not regular for $\kappa \Rightarrow 0$

(20a) RG transformation for the 2d Ising model



triangular lattice

$$l_a = a\sqrt{3}$$

$$H = K \sum_{\langle ij \rangle} S_i S_j + h \sum_i S_i$$

block spins $\quad S_I = \mathrm{sign}\left(S_1^I + S_2^I + S_3^I \right)$

$S_I = 1$	σ_I	$S_I = -1$	σ_I
	↑↑↑		↓↓↓
	↑↑↓		↓↓↑
	↑↓↑		↓↑↓
	↓↑↑		↑↓↓

$$P(\sigma_i, S_I) = \prod_I \delta\left(S_I - \mathrm{sign} \sum_{i \in I} \sigma_i\right)$$

$$\sum_{S_I} P(\sigma_i, S_I) = 1$$

new Hamiltonian $\quad e^{H'(S_I)} = \sum_{\sigma_i \in I} P(\sigma_i, S_I)\, e^{H(\sigma_i)}$