We have seen before that such scaling relation leads to relations between the critical exponents

18b) Correlation function

Spin-spin correlation function

$$G(r_\ell, t_\ell) = \langle S_I S_J \rangle - \langle S_I \rangle \langle S_J \rangle$$

r_ℓ is the distance between blocks I and J in units of a_ℓ $\quad r_\ell = \frac{r}{\ell}$

$r \gg a \qquad r_\ell \gg \ell a$

$$S_I = \frac{1}{|\bar{m}_\ell|} \frac{1}{\ell^d} \sum_{i \in I} S_i$$

$$\bar{m}_\ell = h_\ell \frac{\ell^{-d}}{h} = \ell^{y_h - d}$$

Note that y_h is defined in terms of h_l

$$\Rightarrow G(r_\ell, t_\ell) = \frac{1}{(\ell^d \ell^{y_h - d})^2} \underbrace{\sum_{\substack{i \in I \\ j \in J}} (\langle S_i S_j \rangle - \langle S_i \rangle \langle S_j \rangle)}_{(\ell^d)^2 (\langle S_i S_j \rangle - \langle S_i \rangle \langle S_j \rangle)}$$

(averages are the same for all spins in a block

I and J are far apart

$$= \ell^{2d - 2y_h} G(r, t)$$

$$\Rightarrow G(\tfrac{r}{\ell}, t\ell^{y_t}, h\ell^{y_h}) = \ell^{2(d - y_h)} G(r, t, h)$$

Choose $l = t^{-\frac{1}{y_t}}$

$$\Rightarrow \quad G(r t^{\frac{1}{y_t}}, 1; h t^{-y_h/y_t}) = t^{2(y_h - d)/y_t} G(r,t,h)$$

$G(r,t)$ depends on $\vec{r}$ only through $r t^{1/y_t}$

$$\Rightarrow t^{2(d-y_h)/y_t} = (r y^{1/y_t})^{2(d-y_h)} r^{-2(d-y_h)}$$

$$\Rightarrow G(r,t,h) = \frac{1}{r^{2(d-y_h)}} f(r t^{\frac{1}{y_h}}, h t^{-y_h/y_t})$$

Near T_c

$$G(r,t,\underset{\overset{\shortparallel}{0}}{h}) \sim \frac{1}{r^{d-2+\eta}} e^{-\frac{r}{\xi}}$$

$$\Rightarrow \xi \sim t^{-\frac{1}{y_t}} \quad \Rightarrow \nu = \frac{1}{y_t}$$

$$2(d-y_h) = d-2+\eta$$

- Kadanoff's ideas do not give the value of y_t and y_h
- Generally, the assumption that H_e is of the same form as the original Hamiltonian is not true

19 Renormalization group

19a Introduction

- Do block spin transformation
- Rescale length so that spins are again separated by a
- allow for a general form of the Hamiltonian

$$-\beta H_{\Omega} = \sum \underset{\substack{\uparrow \\ \text{coupling constants}}}{K_n} \; \underset{\substack{\uparrow \\ \text{local operators}}}{\Theta_n} \{S_k\}$$

Under a block spin transformation
$K_n \to K_n'$

We will repeat this operation ∞ many times.
This will be the source of the nonanalyticity

At the critical point, because of the ∞ correlation length, the Hamiltonian (free energy) will be invariant under this transformation

We are looking for the fixed points of this transformation

19b) Properties of the RG transformation

RG transformation $\quad R_\ell(K_i) = K_i'$

- coarse graining into blocks ℓ^d
- rescaling by ℓ

· RG group in a semigroup (the inverse transformation does not exist; information is lost)

$$R_{\ell_1} R_{\ell_2} = R_{\ell_1 \ell_2}$$

Formal definition

$$e^{H_{N'}(K', S_I)} = \mathrm{Tr}_{S_i}\, P(S_i, S_I)\, e^{H_N(K, S_i)}$$

projection operator such that S_I has the same range as S_i

addition requirements
i) $P(S_i, S_I) \geq 0$
ii) $P(S_i, S_I)$ has symmetries of system
iii) $\sum_{S_i} P(S_i, S_I) = 1$

Reasons

for i) we need positive definite Boltzmann weights
for ii) the renormalized Hamiltonian has to be of the same form as the original one
For iii) This is to make sure ~~that~~ the partition function is invariant under the RG transformation

$$Z_{N'}(K') = \mathrm{Tr}_{S_I} e^{H_{N'}(K', S_I)}$$

$$= \mathrm{Tr}_{S_I} \mathrm{Tr}_{s_i} P(s_i, S_I) e^{H_N(K, s_i)}$$

$$= \mathrm{Tr}_{s_i} e^{H_N(K, s_i)} = Z_N(K)$$

Free energy

$$g(K) = \frac{1}{N} \log Z_N(K) = \frac{1}{N} \log Z_{N'}(K')$$

$$= \frac{1}{e^d N'} \log Z_{N'}(K') = \frac{1}{e^d} g(K')$$

This is what we used before.

19c) Origin of singularities

After a finite number of renormalization steps the K' are a regular function of K. A singularity is obtained after ∞ many steps

Example $\quad K_{n+1} = K_n^2$

$K_1 = K$

$K \in [0, 1)$ then $\lim_{n\to\infty} K_n = 0$

$K > 1$ then $\lim_{n\to\infty} K_n = \infty$

Discontinuity for $n \to \infty$

Fixed points $\quad 0, 1, \infty$

basin of attraction of 0 $[0, 1)$

basis of attraction of ∞ $[1, \infty)$

1 is an unstable fixed point

the set of points $K_1, K_2, K_3, \ldots$ is called the renormalization group flow