$$Z = \int D\phi' \, e^{-H_0 - H_{int}}$$

$$= \int D\phi' \, e^{-H_0} \left(1 - H_{int} + \tfrac{1}{2} H_{int}^2 + \cdots\right)$$

expansion diverges for $T \to T_c$ for $d < 4$. For $d > 4$ it becomes a very accurate expansion for $T \to T_c$ with a very small expansion parameter. This does not imply that p.t. converges

$$Z = \int d\phi \, e^{-H_0 - \int d^dx \, \frac{u_0'}{4} \phi^4}$$

This integral diverges if we replace $u_0' \to -u_0'$
$\Rightarrow Z(u_0')$ has an essential singularity at $u_0' = 0$

Generally, p.t. diverges and is at best an asymptotic series

eg $\sum_n n! \, a^n$ converges until $n! \simeq a^n$

$\sum (-1)^n n! \, a^n$ is Borel resummable
$\sum n! \, a^n$ is not

17b) Anomalous dimension

$r_0^{-\frac{1}{2}}$ is the only length scale

$\Rightarrow \xi \sim r_0^{-\frac{1}{2}} \sim t^{-\frac{1}{2}}$
correlation length

However, in general $\xi \sim t^{-\nu}$ with $\nu \neq \frac{1}{2}$

Example 2d Ising model has $\nu = 1$.

How can we reconcile this?

This can only be done if we have another length scale in the problem. Indeed, this scale exists and is the microscopic scale such as the lattice spacing

Then we can have $\xi \sim r_0^{-\frac{1}{2}} f(\underbrace{r_0 a^2}_{\text{dimensionless}})$

If $f(x) \sim x^{\theta}$ for $x \to 0$

then $\xi \sim r_0^{-\frac{1}{2}} (r_0 a^2)^{\theta} \sim r_0^{\theta - \frac{1}{2}} a^{2\theta}$
$\sim t^{\theta - \frac{1}{2}} a^{2\theta}$
for $t \to 0$

then $\nu = \theta - \frac{1}{2}$
θ is called the anomalous dimension

This explains that $a \ll \xi$ can enter in physical observables. $F(\frac{a}{\xi})$ in physical quantities can only be replaced by $F(0)$ if F is regular at 0

If $F(\frac{a}{\xi}) \sim (\frac{a}{\xi})^{\theta}$ this cannot be done and a enters in the observable.

c) Field Theory

- time is replaced by $it = \tau$
  then $ds^2 = d\vec{r}^2 + d\tau^2$

Then there is no difference between space and time. Time is an extra dimension. $d = 4$

In Field theory we do not have an underlying microscopic scale $\Rightarrow$ for

$$H_{eff} = \int \left(\tfrac{1}{2}(\nabla\phi)^2 + \tfrac{1}{2} r_0 \phi^2\right) d^d r$$

$$\langle \phi(r)\,\phi(r') \rangle = \int \frac{d^d k}{(2\pi)^d} \frac{1}{r_0^2 + k^2} e^{i\vec{k}(\vec{r}-\vec{r}')}$$

integral diverges for $d \geq 2$

Quantum field theory limit

$$\lim_{\Lambda \to \infty} \int_{-\Lambda}^{\Lambda} \frac{d^d k}{(2\pi)^d} \frac{1}{r_0^2(\Lambda) + k^2} e^{ik(r-r')}$$

regularization: introduce cutoff so that all integrals are finite

renormalization: adjust $r_0(\Lambda)$ so that the limit $\Lambda \to \infty$ can be taken; if this can be done the theory is said to be renormalizable

18. Kadanoff theory

18 a) Block spins

$$\beta H_\Omega = \underbrace{-\beta J}_{K} \sum_{\langle ij \rangle} S_i S_j - \underbrace{\beta H}_{h} \sum_i S_i$$

Our aim is to study critical exponents and we focus on the calculation of the singular part of the free energy

$$f_s(t,h) \qquad \text{For } T \to T_c$$

For $a \ll la \ll \xi(T)$, the spins inside a block $(la)^d$ are roughly parallel

A Block spin is defined as the average spin of this block

$$S_I = \frac{1}{|\bar{m}_l|} \frac{1}{l^d} \sum_{i \in} S_i$$

$$\bar{m}_l = \frac{1}{l^d} \left| \sum_{i \in I} \langle S_i \rangle \right| \Rightarrow \langle S_I \rangle = \pm 1$$

Hypothesis: the block spins also interact via a nearest neighbor interaction

$$\Rightarrow -\beta H_l = K_l \sum_{\langle IJ \rangle}^{Nl^{-d}} S_I S_J + h_l \sum_{I=1}^{Nl^{-d}} S_I$$

The correlation of length is a fixed length at a given temperature

$$\xi = \xi_\ell a_\ell \Rightarrow \xi_\ell = \frac{\xi_1}{\ell}$$

correlation length in units of the new lattice spacing a_ℓ

But now the correlation length is shorter
$\Rightarrow H_\ell$ is further away from T_c and $|t_\ell| > |t|$

magnetic field $h \sum_i S_i \approx h \sum_I \sum_{i \in I} S_i$

$$= h \sum_I \ell^d |\bar{m}_\ell| \sum_I S_I$$

$$\Rightarrow h_\ell = h \bar{m}_\ell \ell^d$$

Hamiltonian is the same $\Rightarrow$ functional dependence of the free energy is the same

$$\Rightarrow \underset{\#\text{ of links}}{N} f_s(t,h) = \underbrace{N \ell^{-d}}_{\#\text{ of links}} f_s(t_\ell, h_\ell)$$

second assumption $\quad t_\ell = t \ell^{y_t} \quad y^t > 0$

$\qquad h_\ell = h \ell^{y_h} \quad y^h > 0$

Scaling relation $\quad f_s(t,h) = \ell^{-d} f_s(t \ell^{y_t}, h \ell^{y_h})$

ℓ is a free parameter that can be chosen
Then if we choose $\ell = |t|^{-1/y_t}$
we have

$$f_s(t,h) = |t|^{\frac{d}{y_t}} f_s(1, h|t|^{-y_h/y_t})$$