Stationary solution

$$\frac{\partial P}{\partial \tau} = 0 \Rightarrow \partial_y\left(\partial_y P + P\frac{\partial S}{\partial y}\right) = 0$$

$$\Rightarrow P(y) \sim e^{-S(y)}$$

$$\Rightarrow \langle F(q)\rangle = \lim_{T\to\infty}\frac{1}{T}\int_0^T F(q(\tau))$$

if the probability distribution evolves to the stationary solution.

16h) Stability analysis

$P_{eq} = e^{-S(q)}$ substitute $P(y,\tau) = P_{eq}^{\frac{1}{2}}\psi(y,\tau)$

$$\Rightarrow \partial_y P = \tfrac{1}{2}P_{eq}^{-\frac{1}{2}}(\partial_y P_{eq})\psi + P_{eq}^{\frac{1}{2}}\partial_y\psi$$

$$\partial_y^2 P = -\tfrac{1}{4}P_{eq}^{-\frac{3}{2}}(\partial_y P_{eq})^2\psi + \tfrac{1}{2}P_{eq}^{-\frac{1}{2}}\partial_y^2 P_{eq}\psi + 2\tfrac{1}{2}P_{eq}^{-\frac{1}{2}}\partial_y P_{eq}\partial_y\psi + P_{eq}^{\frac{1}{2}}\partial_y^2\psi$$

$$\Rightarrow P_{eq}^{\frac{1}{2}}\partial_\tau\psi = \partial_y P\frac{\partial S}{\partial y} + P\partial_y^2 S + \partial_y^2 P$$

$$= -\tfrac{1}{2}P_{eq}^{\frac{1}{2}}\cancel{\left(\frac{\partial S}{\partial y}\right)^2\psi}\ \cancel{P_{eq}^{\frac{1}{2}}\partial_y\psi\,\partial_y S} + \sqrt{P_{eq}}\,\partial_y^2 S\,\psi$$
$$-\tfrac{1}{4}P_{eq}^{-3/2}\left(\frac{\partial S}{\partial y}\right)^2\psi + \tfrac{1}{2}P_{eq}^{\frac{1}{2}}\left(-\partial_y^2 S + \cancel{(\partial_y S)^2}\right)\psi$$
$$-\cancel{\sqrt{P_{eq}}\,\partial_y S\,\partial_y\psi} + P_{eq}^{\frac{1}{2}}\partial_y^2\psi$$

$$\Rightarrow \partial_\tau\psi = \partial_y^2\psi - \frac{1}{4}\left(\frac{\partial S}{\partial y}\right)^2\psi + \frac{1}{2}\frac{\partial^2 S}{\partial y^2}\psi$$

$$\psi = \psi_n e^{-\lambda_n\tau}$$

$$\Rightarrow -\lambda_n \psi_n = \partial_y^2 \psi_n - \frac{1}{4}\left(\frac{\partial S}{\partial y}\right)^2 \psi_n + \frac{1}{2}\frac{\partial^2 S}{\partial y^2}\psi_n$$

$$\psi_0 = P_{eq}^{\frac{1}{2}} = e^{-S(y)/2} \Rightarrow \lambda_0 = 0$$

$$\partial_y^2 - \frac{1}{4}\left(\frac{\partial S}{\partial y}\right)^2 + \frac{1}{2}\frac{\partial^2 S}{\partial y^2}$$

$$= -\left(\partial_y + \frac{1}{2}\partial_y S\right)\left(\partial_y + \frac{1}{2}\partial_y S\right)^+$$

$$\Rightarrow \text{all } \lambda_n \geq 0 \qquad \left(\partial_y^+ = -\partial_y\right)$$

16i) Microcanonical Simulations

We wish to simulate the distribution $\rho(q) = e^{-\beta S(q)}$
We extend the distribution with a kinetic term

$$\rho(q, \dot{q}) = e^{-\beta\left(S(q) + \frac{\dot{q}^2}{2}\right)}$$

then

$$\rho(q) = \int d\dot{q}\, \rho(q, \dot{q})$$

We use that in the thermodynamic limit the canonical distribution and the microcanonical distribution are equivalent. The latter is

$$\delta\left(E - S(q) - \frac{\dot{q}^2}{2}\right)$$

If the classical motion is ergodic, then the "time" evolution covers the energy-shell uniformely.
So we evolve according to Newton

$$\ddot{q}(t) = -\frac{\partial S}{\partial q}$$

ergodic motion in phase space is expected if the system is sufficiently complex

- β follows from the virial theorem

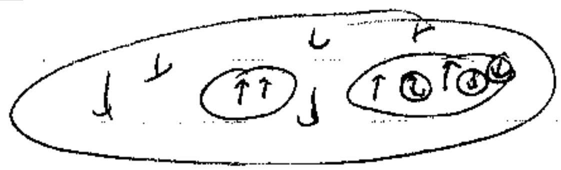
$$\frac{1}{\beta} = \left\langle \frac{\dot{q}^2}{2} \right\rangle$$

$$\Rightarrow \quad \frac{1}{T} \int_0^T F(q)\, d\tau \rightarrow \frac{\int dq\, e^{-\beta S(q)} F(q)}{\int dq\, e^{-\beta S(q)}}$$

16) Critical slowing down

Near the critical point of a second order phase transition the correlation length diverges

Typical configuration



After a couple of updates the configuration will still look more or less the same. Spins like to stay parallel. So we have to do many sweeps before we get a statistically independent configuration.

$$\langle C(t)\, C(t+\bar{t})\rangle \sim e^{-\frac{\bar{t}}{\xi_\tau}}$$

$$\xi_\tau \sim (T-T_c)^{-\varepsilon}$$

This phenomenon is known as critical slowing down

Two problems for $T \to T_c$

i) The size of the lattice has to be much larger than the correlation length

ii) Critical slowing down

17) Anomalous dimensions

see ch 17 of Goldenfeld

Landau-Ginzberg theory

$$Z = \int D\phi \; e^{-H_{eff}} \quad , \quad D\phi = \prod_r d\phi(r)$$

$$H_{eff} = \int d^d r \left(\tfrac{1}{2}(\nabla\phi)^2 + \tfrac{1}{2} r_0 \phi^2 + \tfrac{u_0}{4}\phi^4 \right)$$

$$r_0 = T - T_c = t$$

17a) Dimensional analysis

$$[H_{eff}] = 1$$
$$[\phi] = L^{1-\frac{d}{2}}$$
$$r_0 = L^{-2}$$
$$u_0 = L^{d-2}$$

we define our length scale by $\frac{1}{\sqrt{r_0}}$ and introduce dimensionless variables

Then $$H_{eff} = \int d^d x \left(\underbrace{\tfrac{1}{2}(\nabla\phi')^2 + \tfrac{1}{2}\phi'^2}_{H_0} + \underbrace{\tfrac{1}{4}u_0'\phi'^4}_{H_{int}} \right)$$

$$\phi' = \phi \, r_0^{\frac{1}{2}(1-\frac{d}{2})}$$
$$u_0' = u_0 \, r_0^{\frac{1}{2}(d-4)} = u_0 (T-T_c)^{\frac{d-4}{2}}$$