14e) Mermin Wagner theorem

Long range order is absent in systems with continuous symmetry for $d \leq 2$

Proof $\quad G_\perp(r-r') = \int \frac{d^d k}{(2\pi)^d} \frac{e^{ik(r-r')}}{K^2}$

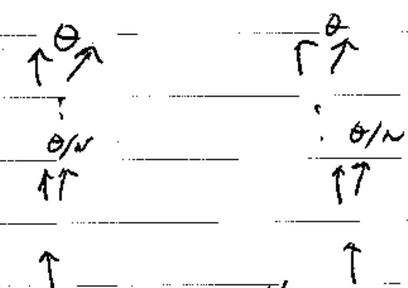
$d \leq 2$ then $G_\perp$ diverges for $k \to 0$

$\Rightarrow \exists$ large transverse fluctuations

$\Rightarrow \langle S \rangle = 0$

Example

$N \uparrow \cdots \cdots \uparrow \qquad E_{vac} = -N^d$



$1 \uparrow \cdots \uparrow$

$$E_{ex} = -N^{d-1} \sum_{i=1}^{N} \cos\frac{\theta}{N}$$

relative weight $\quad e^{-\beta(E_{vac} - E_{ex})}$

$$E_{vac} - E_x = -N^d \left(\cos\frac{\theta}{N} - 1\right) \simeq -N^d \frac{1}{2} \frac{\theta^2}{N^2}$$

$$\Rightarrow e^{-\beta(E_{vac} - E_{ex})} = e^{-\beta \frac{\theta^2}{2} N^{d-2}}$$

$d \leq 2$ Both configuration contribute for $N \to \infty$

$\Rightarrow$ order will be destroyed

15) Kosterlitz-Thouless transition

a)

Continuous symmetries cannot be broken in $d \leq 2$
This does not mean that phase transitions are absent.
Famous example: Kosterlitz-Thouless transition in the $O(n)$ model with $n=2$ and $d=2$

We write the $O(n)$ model in terms of complex fields
$$\phi = S_1 + i S_2$$
$$(\nabla \phi)^2 = (\nabla \vec{S})^2$$

$$\phi = \rho(r)\, e^{i\theta(r)}$$

$$\Rightarrow H = \int d^2r \left(\tfrac{1}{2} |(\nabla \phi)|^2 + \tfrac{u_0}{4}\left(\left(|\phi|^2 + \tfrac{r_0}{u_0} \right)^2 - \underbrace{\tfrac{r_0^2}{u_0}}_{\text{irrelevant constant}} \right) \right)$$

$$\nabla \phi \nabla \phi^* = (\nabla \rho\, e^{i\theta} + \rho\, i \nabla \theta\, e^{i\theta})(\nabla \rho\, e^{-i\theta} - \rho\, i \nabla\theta\, e^{-i\theta})$$
$$= (\nabla \rho)^2 + \rho^2 (\nabla \theta)^2$$

$$\Rightarrow H = \int d^2r\ \tfrac{1}{2}\left((\nabla \rho)^2 + \rho^2 (\nabla \theta)^2 \right) + \tfrac{u_0}{4}\left(\rho^2 + \tfrac{r_0}{u_0} \right)^2$$

The fluctuations of ρ are suppressed by the mass term $\Rightarrow$ we can neglect $(\nabla \rho)^2$

ρ^2 is fixed by the saddle point equation
$$\langle \rho^2 \rangle = -\frac{r_0}{u_0}$$

As Hamiltonian we obtain

$$H = N \int d^2 r \ \tfrac{1}{2} (\nabla \theta)^2 \qquad N = -\frac{r_0}{u_0}$$

- Low temperature phase

in Fourier space $H = N \int \frac{d^2 k}{(2\pi)^2} \ldots \theta^2(k)$

$$\Rightarrow \langle \theta(k)\, \theta(k') \rangle = \frac{1}{\beta N k^2} (2\pi)^2 \delta^2 (k - k')$$

$$(Z = e^{-\beta H})$$

lattice cutoff $\rightarrow \Lambda$

$$\Rightarrow \langle (\theta(r) - \theta(r'))^2 \rangle = \int_0^{\Lambda} \frac{d^2 k_1}{(2\pi)^2} \frac{d^2 k_2}{(2\pi)^2} \langle \theta(k_1)\, \theta(k_2) \rangle$$

$$\times \left(e^{ik_1 r + ik_2 r} - e^{ik_1 r + ik_2 r'} - e^{ik_1 r' + ikr} + e^{ik_1 r' + ik_2 r'} \right)$$

$k_2 = -k_1$

$$= \int_0^{\Lambda} \frac{d^2 k_1}{(2\pi)^2} \frac{(2\pi)^2}{\beta N k^2} \left(2 - 2 \cos \vec{k} (\vec{r} - \vec{r}') \right)$$

$$= \frac{2\pi \times 2}{\beta N} \int_0^{\Lambda} \frac{dk}{k} \left(1 - J_0 (\underbrace{k |\vec{r} - \vec{r}'|}_{\equiv k'}) \right)$$

$J_0(0) = 1$

$$= \frac{4\pi}{\beta N} \int_0 \frac{dk'}{k'} \left(1 - J_0(k') \right)$$

$$\simeq \frac{4\pi}{\beta N} \log (\Lambda |r - r'|)$$

$$= \frac{4\pi k_B T}{N} \log (\Lambda |r - r'|)$$

$$\frac{N}{2}\int (\nabla\theta)^2 d^d r = \frac{N}{2}\sum_{\langle nm\rangle} (\theta_n - \theta_m)^2$$

$$= \frac{N}{2}\sum_{\langle nm\rangle} (1 - 2\cos(\theta_n - \theta_m))$$

$$\Rightarrow \langle e^{i\theta_0} e^{-i\theta_n}\rangle = \frac{1}{Z}\int_0^{2\pi} \prod_m d\theta_m\, e^{i\theta_0 - i\theta_n}$$

$$\times e^{\beta N \sum_{\langle mn\rangle} \cos(\theta_m - \theta_n)}$$

Expand in powers of β

Finite contribution is obtained if all rotating exponents cancel

$$\Rightarrow \langle e^{i\theta_0} e^{-i\theta_n}\rangle = (\beta N)^n = e^{-n \log \frac{1}{\beta N}}$$

$$n = |\vec{n}|$$

correlation function fall off exponentially

- low T : symmetry is not spontaneously broken but correlation function fall off as a power law

- high T : no long range order with exponentially falling correlation function

There should be a transition temperature. This is the Kosterlitz-Thouless temperature T_{KT}

In terms of the original fields the correlation function is given by

$$\langle \phi(r)\,\phi^*(r')\rangle = N^2 \langle e^{i\theta(r)} e^{-i\theta(r')} \rangle$$

$$= \frac{\int \prod_r d\theta(r)\; e^{i\theta(r)-i\theta(r')}\, e^{-\beta N\int d^d r\,(\nabla\theta)^2}}{\int \prod_r d\theta(r)\; e^{-\beta N\int d^d r\,(\nabla\theta)^2}}$$

average over a Gaussian distribution is given by the leading order cumulant expansion

$$= N^2 e^{-\frac{1}{2}\frac{4\pi K_B T}{N}\log r\Lambda} \sim r^{-\eta}$$

$$\eta = \frac{2\pi K_B T}{N}$$

For $n \to \infty$ we have the cluster property

$$\langle e^{i\theta_0} e^{-i\theta_n}\rangle = \langle e^{i\theta_0}\rangle\langle e^{-i\theta_n}\rangle$$

$$\Rightarrow \langle e^{i\theta_0}\rangle = 0$$

- <u>High temperature phase</u>

For $\beta \to 0$ the fluctuations in θ are large we have to take into account that θ is an angle. To do this we use a lattice discretization of the Hamiltonian

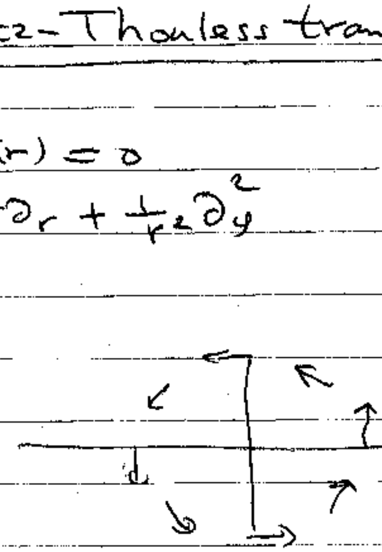
15b) Physical picture of Kosterlitz–Thouless transition

Equations of motion $\vec{\partial}^2 \theta(r) = 0$

$$= \frac{1}{r}\partial_r r \partial_r + \frac{1}{r^2}\partial_\varphi^2$$

Vortex solutions

$$\theta(r) = n\varphi$$



$\theta(r) = \varphi$

Winding number $\oint \vec{\partial}\theta \cdot d\vec{l}$

$$= \int \frac{1}{r}\partial_\varphi \theta \, r d\varphi = 2\pi n$$

Energy of vortex $E = \frac{N}{2} \int (\nabla\theta)^2 dr$

$$= \frac{N}{2} \int \left((\partial_r \theta)^2 + \frac{1}{r^2}(\partial_\varphi \theta)^2 \right) d^2 r$$

$$= \frac{N}{2} n^2 \, 2\pi \int_{\Lambda^{-1}}^{L} \frac{dr}{r} = \frac{N n^2 \pi}{2} \log L\Lambda$$

L is the linear dimension of the system
Λ^{-1} is the lattice cutoff.

At low temperatures, vortices will be present
At high temperature we make an estimate of the free energy

$$F = E - TS$$

entropy of a vortex $\quad S = k_B \log \left(\frac{L}{\Lambda^{-1}}\right)^2$

total # of lattice points

multiplicity of vortex is given by total number of lattice points

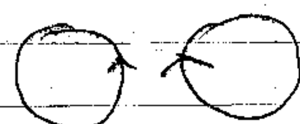
$$\Rightarrow F = \frac{N n^2 \pi}{2} \log L\Lambda - k_B T\, 2 \log L\Lambda$$

free energy is minimized for $n = 1$

Then $\quad T_{KT} = \frac{\pi N}{4 k_B}$

$T > T_{KT}$: it is advantageous to create vortices

$T < T_{KT}$: vortices are bound in vortex-anti-vortex pairs



spins like to be parallel

$E_{pair} \sim 2\pi N \log \frac{r}{\Lambda}$