14 Continuous symmetry

a) Goldstones theorem

If a continuous symmetry is spontaneously broken, the transverse correlations show a power law behavior for $T<T_c$

Mermin - Wagner theorem (Coleman theorem)

Long range order is absent for $d \leq 2$

Exception: Kosterlitz-Thouless theorem in the classical XY model

b) Goldstone's theorem for O(n) model

$$H = \int d^d r \left(\frac{1}{2}(\vec{\nabla} S)^2 + \frac{1}{2} r_0 S^2 + \frac{1}{4} u_0 S^4 - h S \right)$$

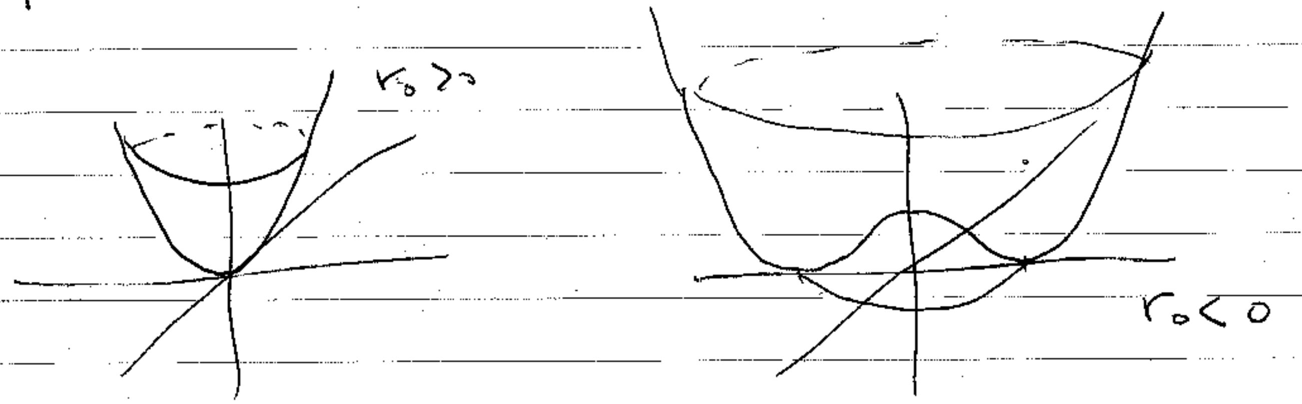
$$S^2 = \sum_{\alpha=1}^{n} S_\alpha^2(r) \qquad (\vec{\nabla} S)^2 = \sum_{i=1}^{d} \sum_{\alpha=1}^{n} (\partial_i S_\alpha)^2$$

$$Z = \int \prod_r \prod_\alpha dS_\alpha(r)\, e^{-\beta H}$$

For $h=0$, H has an $O(n)$ symmetry

invariance $S \to OS \qquad O \in O(n)$

potential $\quad V(s) = \frac{1}{2} r_0 s^2 + \frac{1}{4} u_0 s^4$



for $h \parallel \hat{1}$ axis we have $\hat{m} \parallel \hat{1}$

c) Susceptibilities

longitudinal correlation function

$$\hat{G}_{\parallel}(k) = \langle S_1^2(k) \rangle \equiv \langle S_1(k)\, S_1(-k) \rangle$$

transverse correlation function

$$V \delta_{\vec{k}+\vec{k}'} \delta_{\alpha\beta} \hat{G}_{\perp}(k) = \langle S_\alpha(k)\, S_\beta(k') \rangle_{\alpha,\beta \geq 0}$$

We assume translational invariance

$$\langle S_1(x) S_1(y) \rangle = G_{\parallel}(x-y) = \int \frac{d^d k}{(2\pi)^d} e^{ik(x-y)} \hat{G}_{\parallel}(k)$$

$$= \langle \int \frac{d^d k}{(2\pi)^d} \frac{d^d k'}{(2\pi)^d} e^{ikx + ik'y} \langle S_1(k)\, S_1(k') \rangle \rangle$$

integrate over $\frac{x+y}{2}$ as $\frac{1}{V} \int d^d \frac{x+y}{2}$

this gives factor $(2\pi)^d \delta^d(k+k') \Rightarrow \hat{G}_{\parallel}(k) = \langle S_1(k) S_1(-k) \rangle$

longitudinal susceptibility

$$\chi_{\parallel} = \hat{G}_{\parallel}(0)$$

transverse susceptibility $\chi_{\perp} = \hat{G}_{\perp}(0)$

magnetization $m_\alpha = \frac{1}{\beta}\partial_{h_\alpha} \log Z$

susceptibilities $\chi_{\alpha\beta} = \partial_{h_\beta} m_\alpha = \frac{1}{\beta}\partial_{h_\alpha}\partial_{h_\beta} \log \underbrace{Z}_{e^{-\beta F}}$

O(N) symmetry $\Rightarrow$ Z is only a function of the length of h

$$\Rightarrow \chi_{\alpha\beta} = -\partial_{h_\alpha}\partial_{h_\beta} f(h)$$

$$= -\partial_{h_\alpha} \frac{h_\beta}{h} f'(h)$$

$$= -\left(\frac{\delta_{\alpha\beta}}{h} - \frac{h_\alpha h_\beta}{h^3}\right) f'(h) - \frac{h_\alpha h_\beta}{h^2} f''(h)$$

$h_\alpha \parallel \uparrow \Rightarrow \chi_{\parallel} = -f''(h)$

$\alpha, \beta \geq 2 \Rightarrow \chi_{\perp}(h) = -\frac{1}{h}\frac{\partial f}{\partial h} = -\frac{m}{h}$

Consequence: for long range order we have $m \neq 0$ and $\chi_{\perp}^{-1}(0) = 0$

It takes an infinitesimal amount of magnetic field to rotate the magnetization

Excitations at nonzero momentum can also be split as

- transverse modes or Goldstone modes. Their energy vanishes for $\lambda \to \infty$

- longitudinal modes

14d) Goldstone's theorem

i) Find the minimum of the potential

$$V(S) = \frac{1}{2} r_0 \, S_\alpha S_\alpha + \frac{u_0}{4} (S_\alpha S_\alpha)^4$$

$$\frac{\partial V}{\partial S_\alpha} \Rightarrow \frac{1}{2} r_0 2 S_\alpha + \frac{u_0}{4} 4 S_\alpha^2 S_\alpha$$

$$r_0 > 0 \Rightarrow S_\alpha = 0$$
$$r_0 < 0 \quad \langle S_\alpha^2 \rangle = -\frac{r_0}{u_0}$$

The direction of S_α is determined by the direction of the infinitesimal magnetic field

$$h_\alpha = h n_\alpha = h(1, 0, 0 \ldots 0)$$

$$\Rightarrow S_\alpha = n_\alpha m \qquad m^2 = \frac{-r_0}{u_0}$$

ii) We consider fluctuations about this nonzero mean field

$$S_\alpha = n_\alpha m + m \underbrace{\phi_\alpha(r)}_{\phi_1(r) n_\alpha + \phi_\perp(r)}$$

$$S_\alpha^2 = m^2 + m^2 \phi_1^2 + 2 m^2 \phi_1$$
$$(S_\alpha^2)^2 = m^4 + 2 m^4 \phi_1^2 + 4 m^4 \phi_1 + 4 m^4 \phi_1^2 + O(\phi_1^3)$$

$$\Rightarrow V(S) = \frac{r_0}{2} S_\alpha^2 + \frac{u_0}{4} (S_\alpha^2)^2 = -\frac{1}{4} \frac{r_0^2}{u_0} - \frac{r_0^2}{u_0} \phi_1^2$$

$$\frac{1}{2}(\nabla s)^2 = \frac{m^2}{2}(\nabla \phi_1)^2 + (\nabla \phi_\perp)^2$$

$$\Rightarrow H(\phi) = \frac{m^2}{2}\int d^d r \left((\nabla\phi)^2 + (\nabla \phi_\perp)^2 - 2r_0 \phi_1^2 \right)$$

Fourier transform $\phi_1(r) = \int \frac{d^d k}{(2\pi)^d} e^{ikr} \phi_1(k)$

$$\phi_\perp(r) = \int \frac{d^d k}{(2\pi)^d} e^{ikr} \phi_\perp(k)$$

ϕ_1 real $\Rightarrow$ $\phi_1(k) = \phi_1(-k)$

$\phi_\perp$ " $\Rightarrow$ $\phi_\perp(k) = \phi_\perp(-k)$

$$\Rightarrow H(\phi) = \frac{m^2}{2}\int \frac{d^d k}{(2\pi)^d} \left((k^2 - 2r_0)\phi_1^2(k) + k^2 \phi_\perp^2(k) \right)$$

$$G_{\parallel}(k) = \langle \phi_1(k)\phi_1(k) \rangle = \frac{1}{m^2}\frac{1}{k^2 - 2r_0}$$

$$G_\perp(k) = \langle \phi_{\perp\alpha}(k)\phi_{\perp\beta}(k) \rangle = \frac{\delta_{\alpha\beta}}{m^2}\frac{1}{k^2}$$

$G_\parallel(r)$ decays exponentially

$G_\perp(r)$ decays as a powerlaw $\Rightarrow$ n-1 Goldstone modes

We are interested in the long distance behavior of $G(r)$ $\Rightarrow$ $\frac{1}{k} \gg 1$ $\Rightarrow$ fluctuations in ϕ_1 and $\phi_\perp$ are small $\Rightarrow$ justified to neglect $\mathcal{O}(\phi^3)$ terms