$$x = \frac{n}{N} \qquad \Delta n = 2 \Rightarrow dn = \frac{N}{2}\, dx$$

$$\mathcal{E} = \frac{N}{2} \int_0^1 (\cos k(x) - 1)\, dx$$

$$k(x) = 2\pi x + 2\,\frac{N}{2}\,\frac{1}{N} \int_0^1 \varphi(x, y)\, dy$$

$$2 \cot \varphi(x, y) = \cot \frac{k(x)}{2} - \cot \frac{k(y)}{2}$$

$$\Rightarrow k(x) = 2\pi x + \int_0^1 \cot^{-1}\left(\frac{\cot \frac{k(x)}{2} - \cot \frac{k(y)}{2}}{2} \right) dy$$

$$\Rightarrow \frac{dk(x)}{dx} = 2\pi - \pi + \int \frac{\frac{1}{2} f(\eta) / f(\xi)}{1 + \frac{1}{4}(\xi - \eta)^2}$$

see p. 76a)

$$\frac{dk(x)}{dx} \quad f(\xi) = \frac{2}{1+\xi^2}, \quad \xi = \cot \frac{k(x)}{2} \qquad \frac{d\xi}{dx} = -\frac{1}{f(\xi)} \qquad dy = \frac{dy}{d\eta}\, d\eta$$

$$\eta = \cot \frac{k(y)}{2} \qquad \frac{d\eta}{dy} = -\frac{1}{f(\eta)}$$

$$\Rightarrow \frac{2}{1+\xi^2} = \pi f(\xi) + 2 \int_{-\infty}^{\infty} \frac{f(\eta)\, d\eta}{4 + (\xi - \eta)^2}$$

$$\xi = 0 \Rightarrow 2 = \pi f(0) + 2 \int_{-\infty}^{\infty} \frac{f(\eta)\, d\eta}{4 + \eta^2}$$

Fourier transform $\qquad f(\xi) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} F(k)\, e^{-ik\xi}$

$$\Rightarrow F(k) = \frac{1}{\cosh k}$$

$$\Rightarrow \mathcal{E} = -N \int_0^1 \sin^2 \frac{k(x)}{2}\, dx = -2N \int_{-\infty}^{\infty} d\xi\, \frac{f(\xi)}{1+\xi^2}$$

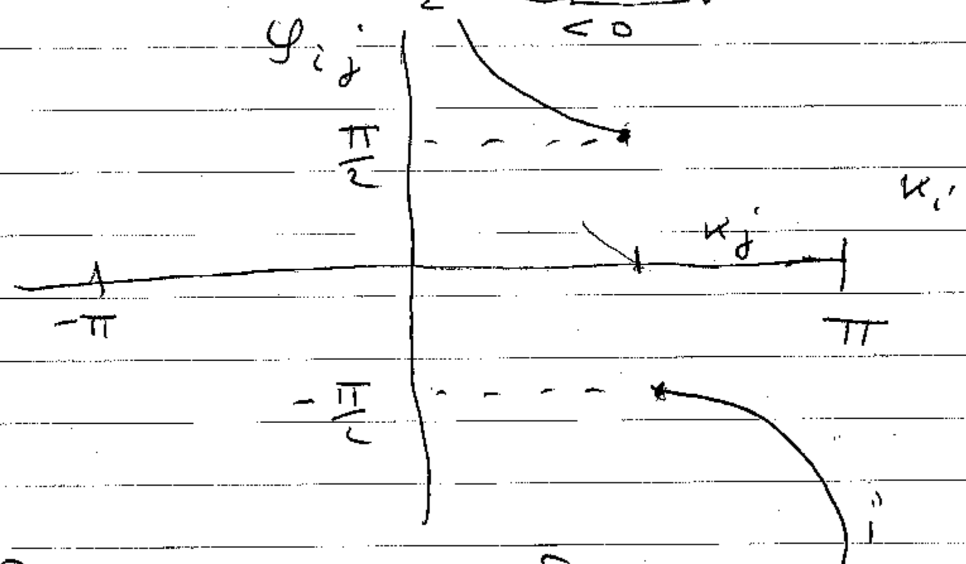
$$= -2N \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, \frac{d\xi}{1+\xi^2}\, e^{-ik\xi} F(k)$$

$$= -N \int_0^{\infty} F(k)\, dk\, e^{-|k|} = -N \log 2 \qquad \text{(Hulthen 1938)}$$

$$K_i = \frac{2\pi n_i - 2\varphi_{ij} - 2\sum_{i,j'} \cdots}{N} \qquad K_j = \frac{2\pi n_j + 2\varphi_{ji} - 2\sum_{i,j'} \cdots}{N}$$

$$2\cot\varphi_{ij} = \cot\frac{k_i}{2} - \cot\frac{k_j}{2}$$

$$\simeq \frac{k_i - k_j}{2} \underbrace{\cot' k_j}_{<0}$$



$$\Rightarrow \partial_x \varphi(x,y) = -\pi\,\delta(x-y)$$

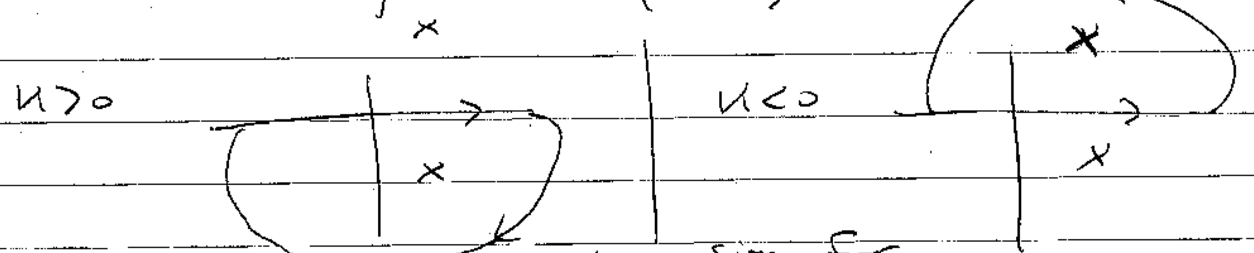
We find

$$\frac{2}{1+\xi^2} = \pi f(\xi) + 2\int_{-\infty}^{\infty} \frac{f(\eta)}{4+(\xi-\eta)^2}$$

$$f(\eta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\kappa\, F(\kappa)\, e^{-i\kappa\eta}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{f(\eta)}{4+(\xi-\eta)^2} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{d\kappa\, d\eta}{2\pi}\, \frac{F(\kappa)\, e^{-i\kappa\eta}}{\left(4+(\xi-\eta)^2\right)}$$

poles: $\eta = \xi \pm 2i$



extra − sign for clockwise contour

$$= \int_{-\infty}^{\infty} \frac{d\kappa}{2\pi}\left(\frac{2\pi i\,\theta(\kappa)(-1)}{(+2i+2i)}\, e^{-i\kappa\xi-2\kappa} + \frac{2\pi i\,\theta(-\kappa)}{-2i-2i}\, e^{-i\kappa\xi}\, e^{+2\kappa}\right) F(\kappa)$$

$$= -\frac{1}{4}\int_{-\infty}^{\infty} d\kappa\, e^{-i\kappa\xi}\left(\theta(\kappa)\, e^{-2\kappa} + \theta(-\kappa)\, e^{2\kappa}\right) F(\kappa)$$

$$\int_{-\infty}^{\infty} d\kappa\, e^{-|\kappa|}\, e^{-i\kappa\xi} = \int_0^{\infty} e^{-\kappa-i\kappa\xi}\, d\kappa + \int_0^{\infty} e^{-\kappa+i\kappa\xi}\, d\kappa$$

$$= \frac{+1}{1+i\xi} + \frac{1}{+1-i\xi} = \frac{2}{1+\xi^2}$$

$$\Rightarrow \int_{-\infty}^{\infty} d\kappa\, e^{-|\kappa|}\, e^{-i\kappa\xi} = \pi\int \frac{d\kappa}{2\pi}\, F(\kappa)\, e^{-i\kappa\xi} + \frac{1}{2}\int_{-\infty} d\kappa\, e^{-i\kappa\xi}\left(\theta(\kappa)\, e^{-2\kappa} + \theta(-\kappa)\, e^{2\kappa}\right) \times F(\kappa)$$

$$\Rightarrow \frac{1}{2}F(\kappa) = e^{-|\kappa|} - \frac{1}{2}\left(\theta(\kappa)\, e^{-2\kappa} + \theta(-\kappa)\, e^{2\kappa}\right) F(\kappa)$$

$K>0$: $\quad F(K) = \dfrac{e^{-|K|}}{\frac{1}{2} + \frac{1}{2} e^{-2K}} = \dfrac{1}{\cosh K}$

$K<0$: $\quad F(K) = \dfrac{e^{+K}}{\frac{1}{2} + \frac{1}{2} e^{2K}} = \dfrac{1}{\cosh K}$

$$\Rightarrow \quad F(K) = \frac{e^{-|K|}}{\frac{1}{2} + \frac{1}{2} e^{-2|K|}} = \frac{1}{\cosh K}$$

Ground state energy

$$\mathcal{E} = -N \int_0^1 \sin^2 \frac{K(x)}{2} \frac{dx}{d\xi} d\xi$$

$$= -N(-) \int_{-\infty}^{\infty} \frac{1}{1+\xi^2} \left(-f(\xi)\right) d\xi$$

$$\cot \frac{K(x)}{2} = \xi \Rightarrow \frac{\cos^2 \frac{K(x)}{2}}{\sin^2 \frac{K(x)}{2}} = \xi^2 \Rightarrow \sin^2 \frac{K(x)}{2} = \frac{1}{1+\xi^2}$$

$$\Rightarrow \mathcal{E} = -N \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} \frac{dK}{2\pi} \frac{F(K) e^{iK\xi}}{1+\xi^2}$$

$$= -N \int_{-\infty}^{\infty} \frac{dK}{2\pi} e^{-|K|} F(K) \frac{2\pi i}{2i}$$

$$= -\frac{N}{2} \int_{-\infty}^{\infty} dK \frac{e^{-2|K|}}{\frac{1}{2} + \frac{1}{2} e^{-2|K|}}$$

$$= -\frac{N}{2} 2 \int_0^{\infty} dK \frac{1}{\frac{1}{2} + \frac{1}{2} e^{2K}} \qquad u = e^{2K}, \quad du = 2u\, dK$$

$$= -N \int_1^{\infty} \frac{du}{2} \frac{1}{u} \frac{1}{\frac{1}{2} + \frac{1}{2}u} = -N \log 2$$