This can be rewritten as

$$2\cos\varphi = \cot\frac{k}{2} - \cot\frac{k'}{2} \qquad \text{(Bethe)}$$

periodic b.c. $\quad f_{j\,i+N} = f_{ij}$
$\qquad i < j < i+N \quad$ (we have $j > i$)

$$\Rightarrow e^{i\varphi}\, e^{i(kj + k'(i+N))} + e^{-i\varphi}\, e^{ik(i+N) + k'j)}$$

$$= e^{i\varphi}\, e^{i(ki + k'j)} + e^{-i\varphi}\, e^{i(k'j + k'i)}$$

coeff of $e^{i(kj + k'i)} \Rightarrow e^{i\varphi + ik'N} = e^{-i\varphi}$

coeff of $e^{i(k'j + ki)} \Rightarrow e^{-i\varphi + ikN} = e^{i\varphi}$

$$\Rightarrow \quad k'N + 2\varphi = 2\pi n$$
$$kN - 2\varphi = 2\pi n$$

if $k' = k \Rightarrow \varphi = \pm\frac{\pi}{2}$ and $f_{ij} = 0$
not a solution

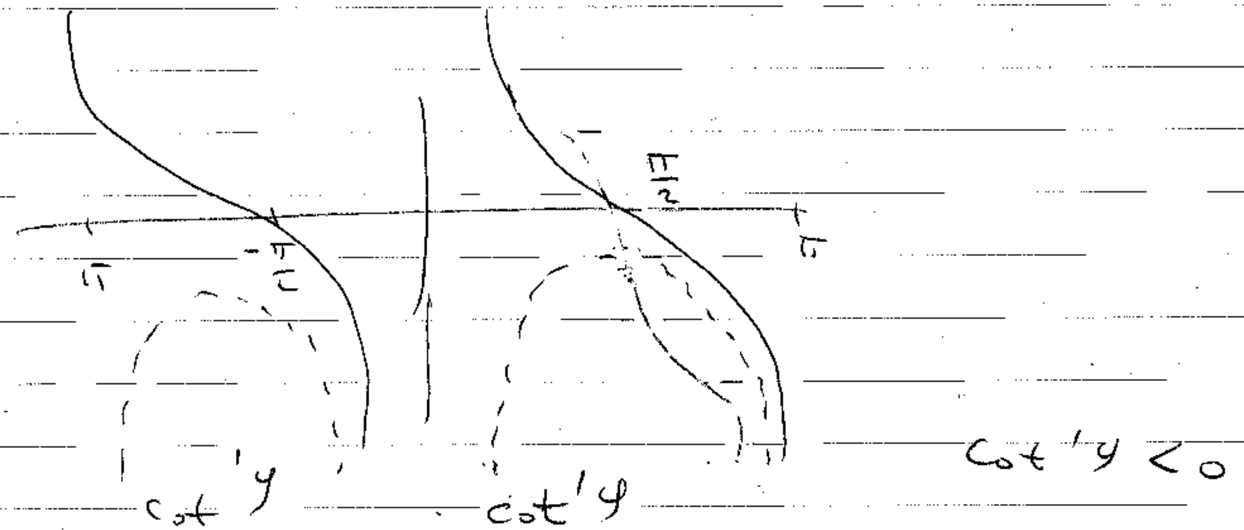
this happens for $\quad n' = n+1$ and $\varphi = \frac{\pi}{2}$
$\qquad n' = n-1 \qquad \varphi = -\frac{\pi}{2}$

φ jumps from $\frac{\pi}{2}$ to $-\frac{\pi}{2}$ going from $n' = n+1$
to $n' = n-1$

$n' = n$ then $2 \cot \varphi = \cot \frac{1}{2}\left(\frac{2\pi n + 2\varphi}{N}\right) - \cot \frac{1}{2}\left(\frac{2\pi n - 2\varphi}{N}\right)$

$$= \cot\left(\frac{\pi n}{N} + \frac{\varphi}{N}\right) - \cot\left(\frac{\pi n}{N} - \frac{\varphi}{N}\right)$$

$$\cong \frac{2\varphi}{N} \cot' \frac{\pi n}{N}$$



$\Rightarrow \exists$ no real solution for φ

$\Rightarrow n' \neq n-1, n, n+1$

d) More spin states

Solution is given by $\Psi = \sum_{i_1 < i_2 < \dots < i_p} f_{i_1 \dots i_p} S^+_{i_1} \dots S^+_{i_p} |0\rangle$

with $f_{i_1 \dots i_p} = \sum_{\substack{\text{permutations} \\ \text{of } \{1,\dots,p\}}} e^{i\left(k_{\pi(1)} i_1 + \dots + k_{\pi(p)} i_p + \sum_{i<j} \varphi_{\pi(i)\pi(j)}\right)}$

$$\varphi_{ij} = -\varphi_{ji}$$

$$2 \cot \varphi_{ij} = \cot \frac{k_i}{2} - \cot \frac{k_j}{2}$$

periodic boundary conditions

$$k_{\pi(i)} N + 2 \sum_{j \neq i}^{p} \varphi_{\pi(i)\pi(j)} = 2\pi\, q_{\pi(i)}$$

$$q_i = 0, \pm 1, \dots \pm \frac{N}{2}$$

$$E = \frac{N}{4} + \sum_{i=1}^{p} \cos k_p - 1$$

If $k_i = k_j$ then $\varphi_{ij} = \pm \frac{\pi}{2}$ $\quad \varphi_{ij} = -\varphi_{ji}$

for each permutation with $\pi(i') = i$ and $\pi(j') = j$
we have a permutation with $\pi(i') = j$ and $\pi(j') = i$

$$\Rightarrow f_{i_1 \dots i_p} = \sum e^{i(\ \ + k_i i_{i'} + \dots + k_j i_{j'}} + i \sum \dots + \varphi_{ij}$$
$$+ e^{i(\dots + k_j i_{i'} + \dots + k_i i_{j'})} + i \sum \dots + \varphi_{ji}$$

$$= 0$$

$\Rightarrow$ all k_i should be different

$$k_i N + 2 \sum_{i'<j'} \cdots + 2\vartheta_{ij} = 2\pi q_i$$

$$k_j N + 2 \sum_{i'<j'} \cdots + 2\vartheta_{ji} = 2\pi q_j$$

$$\left.\begin{array}{l}\vartheta_{ij} = \frac{\pi}{2} \quad \vartheta_{ji} = -\frac{\pi}{2} \\ k_i = k_j\end{array}\right\} \Rightarrow q_j = q_i + 1$$

$\Rightarrow q_j = q_i + 1$ is not allowed

for $\vartheta_{ij} = -\frac{\pi}{2}$, $\vartheta_{ji} = \frac{\pi}{2}$

$\Rightarrow q_j = q_i - 1$ is not allowed

13c) Ground state of the Heisenberg model

$n = 0 \Rightarrow p = \frac{N}{2}$ in our general solution

Take $N = eV$

if $k_i = k_j \Rightarrow \vartheta_{ij} = \pm\frac{\pi}{2} \Rightarrow f_{i_1 \ldots i_p} = 0$

$\Rightarrow$ all k_i are different

$n' = n+1, n, n-1$ is not allowed

$\Rightarrow$

ground state $\qquad n = 1, 3, \ldots N-1$

$$E = \sum_{n\,\mathrm{odd}} (\cos(k_n) - 1) \qquad k_n = \frac{2\pi n + \sum_j \vartheta_{nj}}{N}$$