13. Heisenberg model

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a) Intro

$$H = \sum_{i=1}^{N} \tfrac{1}{2}\left(S_i^+ S_{i+1}^- + S_{i+1}^+ S_i^- + S_i^z S_{i+1}^z \right.$$

spin $\frac{1}{2}$

Theorem Lieb-Mattis J. Math. Phys. 3 (1962) 749

The ground state of H is an angular momentum singlet $J=0$

We wish to find the ground state of H

$J=0 \Rightarrow M=0 \Rightarrow$ groundstate has $\frac{N}{2}$ spins up and $\frac{N}{2}$ spins down

Notice that the total number of spins up is a good quantum number $([J_z, H]=0)$

"Vacuum" all spins down $E = \sum S_i^z S_{i+1}^z = \frac{N}{4}$

one magnon states $\Psi_k = \frac{1}{\sqrt{N}} \sum_i e^{ikR_i} S_i^+ |0\rangle$

$$E = \cos ka \underbrace{- \tfrac{1}{2} 2 + \tfrac{N}{4}}_{\text{from } S_i^z S_{i+1}^z \text{ term}}$$

Anisotropic Heisenberg model $\quad S_{N+1} = S_1$

$$H = \sum_i \tfrac{1}{2}(S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+) + \sum \alpha S_i^z S_{i+1}^z$$

$\alpha = 0$: XY model $\qquad \alpha = \infty$ Ising model

$\alpha = 1$: Heisenberg model

"vacuum" $\quad |0\rangle = |\downarrow\downarrow \cdots \downarrow\rangle$

b) <u>One particle states</u>

We will give a more systematic way of deriving these states instead of guessing the answer

$$\psi = \sum_k f_k S_k^+ |0\rangle$$

$$H\psi = \sum \tfrac{1}{2}(f_{i+1} S_i^+ + f_i S_{i+1}^+)|0\rangle + \alpha\left(\tfrac{N}{4} - \tfrac{2}{2}\right) f_i S_i^+ |0\rangle$$

$H\psi = E\psi \qquad$ equate coeff of $S_i^+|0\rangle$

$$\Rightarrow \tfrac{1}{2} f_{i+1} + \tfrac{1}{2} f_{i-1} + \alpha\left(\tfrac{N}{4} - 1\right) f_i = E f_i$$

$$\varepsilon \equiv E - \frac{N\alpha}{4}$$

Solution $\quad f_n = e^{ikn} \qquad \varepsilon = \cos k - \alpha$

c) Two particle states

Bethe ansatz : $\psi = \sum_{\substack{ij \\ i<j}} f_{ij} S_i^+ S_j^+ |0\rangle$

Distinguish case that i and j are neighbors and not

$$H\psi = \sum_k \tfrac{1}{2}\left(S_k^+ S_{k+1}^- + S_k^- S_{k+1}^+ + 2\alpha S_k^z S_{k+1}^z\right) \sum_{ij} f_{ij} S_i^+ S_j^+ |0\rangle$$

$$= \frac{1}{2} \sum_{\substack{i<j \\ \text{not neighborgs}}} \left(S_{j-1}^+ S_i^+ + S_{i-1}^+ S_j^+ + S_{i+1}^+ S_j^+ + S_{j+1}^+ S_i^+ \right) f_{ij} |0\rangle$$

$$+ \frac{1}{2} \sum_i \left(S_{i+1}^+ S_{i-1}^+ + S_i^+ S_{i+2}^+ \right) f_{i\,i+1} |0\rangle$$

$$- \alpha \frac{1}{4}\, 4 \sum_{\substack{i<j \\ \text{not neighbors}}} f_{ij}\, S_i^+ S_j^+ |0\rangle$$

$$- \alpha \frac{1}{4}\, 2 \sum_i f_i f_{i+1}\, S_i^+ S_{i+1}^+ |0\rangle$$

$$= \mathcal{E} \sum f_{ij}\, S_i^+ S_j^+ |0\rangle$$

coefficient of $S_i^+ S_{i+1}^+ |0\rangle$

$$\tfrac{1}{2}\left(f_{i\,i+2} + f_{i-1\,i+1}\right) - \tfrac{1}{2}\alpha f_{i\,i+1} = \mathcal{E} f_{i\,i+1}$$

coefficient of $S_i^+ S_j^+ |0\rangle$ $\quad j \neq i+1$

$$\tfrac{1}{2}\left(f_{i+1\,j} + f_{i\,j+1} + f_{i\,j-1} + f_{i-1\,j}\right) - \alpha f_{ij} = \mathcal{E} f_{ij}$$

It is easy to solve the 2^{nd} equation

$$f_{mn} = a_1 e^{ikm + ik'n} + a_2 e^{ikn + ik'm}$$

$$\Rightarrow \quad \varepsilon = \cos k + \cos k' - 2\alpha$$

We still have to satisfy the first equation

f_{mn} satisfies 2^{nd} eq for $j = i+1$
substract first equation

$$\Rightarrow \quad \tfrac{1}{2}(f_{ii} + f_{i+1\,i+1}) - \tfrac{1}{2}\alpha\, f_{i\,i+1} = 0$$

Put $a_1 = e^{i\varphi}$ $\quad a_2 = e^{-i\varphi}$

$$\Rightarrow \quad e^{i\varphi}\left(\tfrac{1}{2} + \tfrac{1}{2} e^{ik + ik'} - \alpha\, e^{ik'}\right) + e^{-i\varphi}\left(\tfrac{1}{2}(1 + e^{ik+ik'}) - \alpha\, e^{ik}\right) = 0$$

$\alpha = 0 \Rightarrow \varphi = \frac{\pi}{2}$ o.k. for XY model

$$\alpha = 1 \Rightarrow \cos\varphi\left(1 + e^{i(k+k')} - (e^{ik} + e^{ik'})\right) + i\sin\varphi\,(e^{ik} - e^{ik'}) = 0$$

$$\Rightarrow \cot\varphi = \frac{i(e^{ik} - e^{ik'})}{(1 + e^{i(k+k')}) - e^{ik} - e^{ik'}}$$

$$= \frac{\sin(k-k')/2}{2\cos(k+k')/2 - \cos(k-k')/2}$$