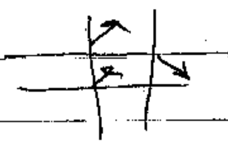
12) XY model

a) Heisenberg model $H = \sum_{\langle ij \rangle} J_{ij} \vec{S}_i \cdot \vec{S}_j$ $J_{ij} > 0$

$\vec{S} = (\sigma_x, \sigma_y, \sigma_z)$ anti-ferromagnet



spin operators are allowed to move in 3 d

XY model: spin operators can only rotate in 2d

$$H = \sum_{\langle ij \rangle} J_{ij} \big(\underbrace{S_i^x S_j^x + S_i^y S_j^y}_{} \big)$$

$$= \tfrac{1}{2}(S_i^x + i S_i^y)(S_j^x - i S_j^y) + \tfrac{1}{2}(S_i^x - i S_i^y)(S_j^x + i S_j^y)$$

$$\Rightarrow H = \sum_{\langle ij \rangle} J_{ij} \left(S_i^+ S_j^- + h.c \right)$$

1 d XY model $H = \frac{1}{2} \sum_{i=1}^{N} \left(S_i^+ S_{i+1}^- + h.c. \right)$

$S_{N+1} = S_1$

all coupling constants
are the same

b) Solution of XY model

two spin system

$$H = \vec{S}_1 \cdot \vec{S}_2 = \frac{1}{2}\left(S_1^+ S_2^- + S_1^- S_2^+\right) + \underbrace{S_{1z} S_{2z}}_{}$$

for Heisenberg model

		E
eigenstates of $H - S_{1z}S_{2z}$	$\lvert\uparrow\uparrow\rangle$	$\frac{1}{4} - \frac{1}{2}\frac{1}{2} = 0$
	$\lvert\downarrow\downarrow\rangle$	$\frac{1}{4} - (-\frac{1}{2})(-\frac{1}{2}) = 0$
	$\frac{1}{\sqrt{2}}\left(\lvert\downarrow\uparrow\rangle + \lvert\uparrow\downarrow\rangle\right)$	$\frac{1}{4} + \frac{1}{2}\frac{1}{2} = \frac{1}{2}$
	$\frac{1}{\sqrt{2}}\left(\lvert\downarrow\uparrow\rangle - \lvert\uparrow\downarrow\rangle\right)$	$-\frac{3}{4} + \frac{1}{4} = -\frac{1}{2}$

the last state is the ground state; because of the antiferromagnetic nature, it has a nontrivial structure

construction of eigenstates

"Vacuum" state: all spins down (not the lowest energy state)

$$S^{-}|\psi\rangle = 0$$

$$|0\rangle = |\downarrow \cdots \downarrow\rangle$$

$$H|0\rangle = 0$$

"one-particle" states : N-1 spins up
1 down

<u>Ansatz</u>: $\Psi_k = \frac{1}{\sqrt{N}} \sum_i e^{i\vec{k}\cdot\vec{R}_i} S_i^+ |0\rangle$

$R_i = i a$ in 1d $\quad \overset{i}{\bullet} \quad \overset{i+1}{\bullet} \quad \overset{i+2}{\bullet}$

$$H\Psi_k = \frac{1}{\sqrt{N}} \sum_n e^{ikna} \frac{1}{2} \sum_j \left(S_j^+ S_{j+1}^- + S_{j+1}^+ S_j^-\right) S_n^+ |0\rangle$$

$$= \frac{1}{\sqrt{N}} \sum_n e^{ikna} \frac{1}{2} \left(S_{n-1}^+ + S_{n+1}^+\right) |0\rangle$$

$$= \frac{1}{\sqrt{N}} \frac{1}{2} \left(e^{ika} + e^{-ika}\right) \sum_n S_n^+ \underset{e^{ikna}}{\wedge} |0\rangle$$

$$= \cos ka \, \Psi_k$$

periodic boundary conditions: $S_{N+1} = S_1$

$$\Rightarrow e^{ikNa} = 1$$

$$\Rightarrow k = \frac{2\pi m}{N}, \quad m \in \mathbb{Z}$$

$$S_i^+ |0\rangle = \frac{1}{\sqrt{N}} \sum_k e^{-ikR_i} \Psi_k$$

inverse Fourier transform

<u>two spin up states</u>

complication $\quad \{S_i^+, S_i^-\} = 1 \qquad [S_i^\pm, S_j^\pm] = 0, \ i \neq j$

transform to different operators that all satisfy anticommutation relations

New operators
$$C_m = (-1)^{\sum_{j<m} n_j} S_m^-$$
$$C_m^* = (-1)^{\sum_{j<m} n_j} S_m^+$$

$$[S_m^-, n_m] = n_m$$

$n < m$

$$\Rightarrow \quad C_m C_n^* + C_n C_m^* =$$
$$(-1)^{\sum_{j<m} n_j} S_m^- (-1)^{\sum_{j<n} n_j} S_n^+ + (-1)^{\sum_{j<n} n_j} S_n^- (-1)^{\sum_{j<m} n_j} S_m^-$$
$$= (-1)^{\sum_{j<m} n_j + \sum_{j<n} n_j} S_m^- S_n^+ + (-1)^{\sum_{j<n} n_j + \sum_{j<m} n_j + 1} S_n^- S_m^-$$
$$= [S_m^-, S_n^+] = 0 \quad \text{because } n \neq m$$

Exercise: Show that $\{C_n, C_n\} = 0$

boundary condition $\quad C_{N+1} = c_1 (-1)^{\sum_{j=1}^{N} n_j + 1}$

The next step is to perform a unitary transformation of on the C_n

$$C_m = \frac{e^{-i\pi/4}}{\sqrt{N}} \sum_q e^{iqn} c_q$$

$$C_{N+1} = c_1 (-1)^{\sum_{j=1}^{N} n_j}$$

This does not change the commutation relations.

boundary conditions

$$\frac{e^{-\pi i/4}}{\sqrt{N}} \sum_q e^{iq(N+1)} c_q = \frac{e^{-\pi i/4}}{\sqrt{N}} \sum e^{iq} c_q (-1)^{\sum_{j=1}^{N} n_j + 1}$$

$$\Rightarrow \quad e^{iqN} = (-1)^{\sum_{j=1}^{N} n_j + 1} \qquad \forall q$$

$$q = \frac{2\pi(n+\frac{1}{2})}{N} \qquad \text{for } \sum n_j \text{ even}$$

$$= \frac{2\pi n}{N} \qquad \text{for } \sum n_j \text{ odd}$$

$$n \in \mathbb{Z}$$

Hamiltonian $H = \frac{1}{2} \sum_j (S_j^+ S_{j+1}^- + S_{j+1}^+ S_j^-)$

$$= \frac{1}{2} \sum_j (-1)^{\sum_{j'<j} n_{j'}} (-1)^{\sum_{j'<j} n_{j'}} S_j^+ S_{j+1}^-$$

$n_j = 0$ otherwise S_j^+ gives zero
extend sum to $j' < j+1$

$$+ (-1)^{\sum_{j'<j+1} n_{j'}} (-1)^{\sum_{j'<j+1} n_{j'}} S_{j+1}^+ S_j^-$$

$n_j = 0$ after S_j^-
$\Rightarrow$ we can restrict sum to $j' < j$

$$= \frac{1}{2} \sum_j c_j^* c_{j+1} + c_{j+1}^* c_j$$

$$= \frac{1}{2} \sum_{j=1}^{N} \frac{1}{\sqrt{N}\sqrt{N}} \sum_{qq'} e^{-iqj} e^{iq'(j+1)} c_q^* c_{q'} + \text{h.c.}$$

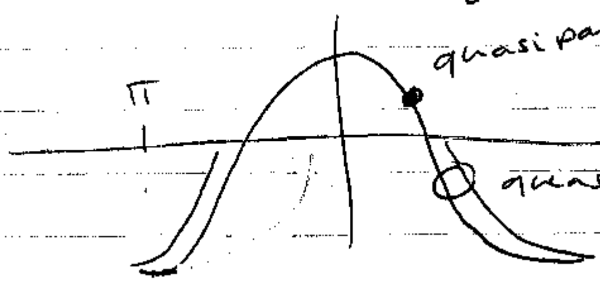
$$= \frac{1}{2} \sum_q \left[e^{iq} c_q^* c_q + e^{-iq} c_q^* c_q \right]$$

$$= \sum_q \cos q \; c_q^* c_q$$

Eigenstates are given by many particle states of free Fermions

one particle excitations $E = \cos q$

a particle is a spinwave or magnon



the state with the lowest energy is the state with all negative energy states filled.

We have a nontrivial vacuum

Many particle states $E = \sum_i \cos q_i$

Dispersion relation for quasiparticles

$\varepsilon(q) = |\sin q|$