$$\cosh \varepsilon_q \doteq \cosh 2K \cosh 2\theta - \cos q \sinh 2K \sinh 2\theta$$
$$> \cosh 2K \cosh 2\theta - \sinh 2K \sinh 2\theta$$
$$= \cosh(2K - 2\theta) > 1$$

$\Rightarrow$ all $\varepsilon_q > 0 \Rightarrow$ The largest eigenvalue of $V(q)$ occurs for all $n_q = 0$

then $$\lambda_{max} = \prod_{q>0} e^{2K\cos q + \varepsilon_q} \times (e^{2K} - e^{-2K})^{\frac{n}{2}}$$

$$\sum n_q = 0 \quad \Rightarrow \quad q = \pm\frac{\pi}{n}, \pm\frac{3\pi}{n}, \ldots, \pm\frac{n-1}{n}\pi$$

$$\Rightarrow \sum \cos q = \sum_{K=1,\ldots n-1}' \cos\frac{\pi K}{n} \doteq 0$$

$$\Rightarrow \lambda_{max} = (2\sinh 2K)^{\frac{n}{2}} e^{\sum_{q>0} \varepsilon_q}$$

$\Delta q = \frac{2\pi}{n}$ ok

$$\sum_{q>0} \varepsilon_q \doteq \frac{1}{2}\int_{-\pi}^{\pi} \frac{n}{2\pi}\, dq\, \varepsilon_q$$

e) Final result and discussion

identity $\quad \varepsilon_q = \frac{1}{2\pi}\int_0^{2\pi} dt \log\left(2\cosh\varepsilon_q - 2\cos t\right)$

remind that $\cosh\varepsilon_q = \cosh 2K \cosh 2\theta - \cos q \sinh 2K \sinh 2\theta$

$$\sinh\theta = \frac{e^{-K}}{\sqrt{e^{2K}-e^{-2K}}} \qquad \cosh\theta = \frac{e^{K}}{\sqrt{e^{2K}-e^{-2K}}}$$

$$\Rightarrow \sinh 2\theta = 2\sinh\theta\cosh\theta = \frac{2e^{-2K}}{1-e^{-4K}} = \frac{1}{\sinh 2K}$$

$$\cosh 2\theta = \frac{\cosh 2K}{\sinh 2K}$$

$$\Rightarrow \cosh\varepsilon_q = \frac{\cosh^2 2K}{\sinh 2K} - \cos q$$

$$\Rightarrow \varepsilon_q = \frac{1}{2\pi}\int_0^{2\pi} dt \log\left(2\frac{\cosh^2 2K}{\sinh 2K} - 2\cos q - 2\cos t\right)$$

$$= \log\frac{2}{\sinh 2K} + \frac{1}{2\pi}\int_0^{2\pi} dt \log\left(\cosh^2 2K - (\cos q + \cos t)\sinh 2K\right)$$

$$\Rightarrow \lambda_{max} = e^{\frac{n}{2}\log 2\sinh 2K + \frac{n}{2}\log\frac{2}{\sinh 2K}} \times e^{\frac{n}{8\pi^2}\int_0^{2\pi} dq \int_0^{2\pi} dt \log\left(\cosh^2 2K - (\cos q + \cos t)\sinh 2K\right)}$$

$$= e^{n\log 2 + \frac{n}{8\pi^2}\int_0^{2\pi} dq \int_0^{2\pi} dt \log\left(\cosh^2 2K - (\cos q + \cos t)\sinh 2K\right)}$$

limit $T \to 0$ then $K \to \infty \longrightarrow \cosh^2 2K \to \frac{e^{4K}}{4}$

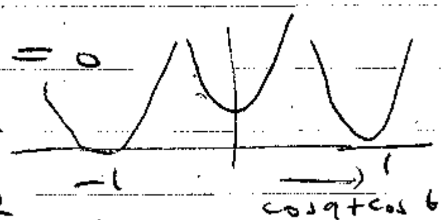
$$\Rightarrow \lambda_{max} = e^{2Kn}$$

$\Rightarrow$ all spins are up

critical temperature

The critical temperature is where the free energy has a singularity. This is where the argument of the logarithm vanishes

$$\cosh^2 2K - (\cos q + \cos t)\sinh 2K = 0$$



$$\overset{1+\sinh^2 2K}{=} \left(\sinh 2K - \frac{\cos q + \cos t}{2}\right)^2 + 1 - \frac{(\cos q + \cos t)^2}{4}$$

Discriminant $D = (\cos q + \cos t)^2 - 4 \Rightarrow D \leq 0$

$$D = 0 \Rightarrow \cos q + \cos t = \pm 2$$

$$\Rightarrow (\sinh 2K \mp 1)^2 = 0$$

$$\Rightarrow \sinh 2K_c = 1$$

critical exponent α

$$Z = e^{-\beta f N^2} \qquad u = \frac{\partial \beta f}{\partial \beta} = J \frac{\partial \beta f}{\partial K}$$

$$\beta f = -\log 2 - \frac{1}{8\pi^2}\int_0^{2\pi} dq \int_0^{2\pi} dt \log\left(\cosh^2 2K - (\cos q + \cos t)\sinh 2K\right)$$

$$\Rightarrow u = -\frac{J}{8\pi^2}\int_0^{2\pi} dq \int_0^{2\pi} dt \, \frac{2.2\cosh 2K \sinh 2K - 2(\cos q + \cos t)\cosh 2K}{\cosh^2 2K - (\cos q + \cos t)\sinh 2K}$$

This can be rewritten as an elliptic integral

$$\Rightarrow u = -J \cosh 2K \left(1 + \frac{2}{\pi}(2\tanh^2 2K - 1)\, K_1(k)\right)$$

$$k = \frac{2\sinh 2K}{\cosh^2 2K}$$

$$K_1(k) = \int_0^{\frac{\pi}{2}} (1 - k^2 \sin^2 \varphi)^{-\frac{1}{2}} \, d\varphi$$

elliptic integral

In order to find the critical exponent we expand around the critical point

$$\sinh 2K = 1 - \alpha \Rightarrow \frac{\cosh^2 2K}{\sinh 2K} = \frac{1 + (1-\alpha)^2}{1-\alpha}$$

$$\beta f = -\log \lambda_{max}^{1/n}$$

$2\pi \to \pi$ gives extra overall factor 4

$$= -\log\left[2\,(1-\alpha)^{\frac{1}{2}}\right] \mp \frac{1}{2\pi^2} \int_0^{\pi} dq \int_0^{\pi} dt \, \log\left(\frac{t^2}{1-t} + 2 - \cos q - \cos t\right)$$

singular behavior comes from $\cos q + \cos t = 2$

expand $\cos q = 1 - \frac{1}{2} q^2$

$\cos t = 1 - \frac{1}{2} t^2$

$$= -\log\left(2(1-\alpha)^{\frac{1}{2}}\right] \mp \frac{1}{2\pi^2} \int_0^{\pi} dq \int_0^{\pi} dt \log\left(\frac{\alpha^2}{1-\alpha} + \frac{q^2 + t^2}{2}\right)$$

singular behavior for $\alpha \to 0$ comes from

$$-\frac{1}{2\pi^2} \int_0^{a} r\, dr \, \log\left(\frac{\alpha^2}{1-\alpha} + \frac{r^2}{2}\right)$$

$$= -\frac{1}{2\pi^2} \frac{1}{2} \left[\left(\frac{\alpha^2}{1-\alpha} + \frac{r^2}{2}\right) \log\left(\frac{\alpha^2}{1-\alpha} + \frac{r^2}{2}\right) - \frac{\alpha^2}{1-\alpha} + \frac{r^2}{2}\right]_0^a$$

$$\sim \frac{1}{4\pi^2} \frac{\alpha^2}{1-\alpha} \log \frac{\alpha^2}{1-\alpha} - \frac{\alpha^2}{1-\alpha}$$

$$\Rightarrow \text{sing } \beta f = \frac{2\alpha^2}{4\pi^2} \log \alpha$$

specific heat $\quad C \sim \partial_t^2 f \sim -\log t$

$$\Rightarrow \alpha = 0$$

critical exponent for magnetization

$$m \sim (T-T_c)^\beta$$

has been calculated analytically by Yang
The calculation is much more complicated

Result : $\beta = \frac{1}{8}$