$$\underset{\sum n_j}{\overset{\parallel}{\sum n_q}} \text{ even} \Rightarrow q = \pm\frac{\pi}{n}, \ldots, \frac{n-1}{n}\pi$$

$$\sum n_q \text{ odd} \Rightarrow q = 0, \pm\frac{2\pi}{n}, \ldots, \pm\frac{(n-2)\pi}{n}, \pi$$

Example $n = 4$

$\sum n_q$ even $\quad q = \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}$

$\sum n_q$ odd $\quad q = 0, \pm\frac{\pi}{4}, \pi$

we have 4 momenta with $\sum n_q$ even $\Rightarrow$ we can occupy, 0, 2 or 4 states

$$\left.\begin{array}{ll} 0 & 1 \\ 2 & 6 = \binom{4}{2} \\ 4 & 1 \end{array}\right\} \text{total } 8$$

$\sum n_q$ odd we can occupy 1 or 3 states

$$\left.\begin{array}{ll} 1 & 4 = \binom{4}{1} \\ 3 & 4 = \binom{4}{3} \end{array}\right\} \text{total } 8$$

$\Rightarrow$ in total we have 16 states, agrees with $2^{n=4}$

$\Rightarrow$ V has 16 eigenvalues.

Lecture # 13

$$c_m = \frac{e^{-\pi i/4}}{\sqrt{n}} \sum_q e^{-iqm} C_q$$

$$V_2 = \prod_{q>0} e^{2K(n_q + n_{-q})\cos q + 2K\sin q (C_q C_{-q} + C_{-q}^* C_q^*)}$$

$$V_1 = (e^{2K} - e^{-2K})^{n/2} \prod_{q>0} e^{-2\Theta(n_q + n_{-q} - 1)}$$

$$\cosh\Theta = \frac{e^{K}}{\sqrt{e^{2K} - e^{-2K}}}$$

$$V(q) = V_1(q) V_2(q)$$

Today discussion of Onsager solution

Note that V_1 and V_2 do not commute. It is simplest to use the tensor representation of V_1 and V_2, and diagonalize the product with Mathematica

```
In[∘]:= n = {{1, 0}, {0, 0}}; c = {{0, 0}, {1, 0}};
        cs = {{0, 1}, {0, 0}};
        on = {{1, 0}, {0, 1}};

In[∘]:= Clear[K];
        m1 = ArrayFlatten[TensorProduct[n, on]] +
           ArrayFlatten[TensorProduct[on, n]] - ArrayFlatten[TensorProduct[on, on]];
        V1 = MatrixExp[2 th m1];
        m2 =
          2 K sq (ArrayFlatten[TensorProduct[c, c]] + ArrayFlatten[TensorProduct[cs, cs]]) +
           2 K cq (ArrayFlatten[TensorProduct[n, on]] + ArrayFlatten[TensorProduct[on, n]]);
        V2 = MatrixExp[ m2];
        ev = Assuming[cq^2 + sq^2 == 1, Simplify[Eigenvalues[V1.V2]]];
        FullSimplify[ev[[3]] + ev[[4]]]
```

Out[∘]= $2\, e^{2\, cq\, K} \left(\mathrm{Cosh}[2\, K]\, \mathrm{Cosh}[2\, th] + cq\, \mathrm{Sinh}[2\, K]\, \mathrm{Sinh}[2\, th]\right)$